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Lyapunov functions and $\mathcal{L}_2$ gain bounds for systems with slope restricted nonlinearities

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ABSTRACT

The stability and $\mathcal{L}_2$ performance analysis of systems consisting of an interconnection of a linear-time-invariant (LTI) system and a static nonlinear element which is Lipschitz, slope restricted and sector bounded is revisited. The main thrust of the paper is to improve and extend an existing result in the literature to enable (i) concise and correct conditions for asymptotic stability of the interconnection and (ii) reasonably tight bounds on the $\mathcal{L}_2$ gain between an exogenous input and a given output to be obtained. Numerical examples indicate that the proposed algorithm performs well compared to competing results in the literature.

1. Introduction

This paper considers the system depicted in Fig. 1 (overleaf) where the nonlinearity is a sector bounded, slope restricted nonlinearity. The problem addressed is:-

Problem 1.

(1) When $w(t) \equiv 0$, find Lyapunov-based conditions which enable global asymptotic stability of the origin of the interconnection to be ascertained.
(2) When $w(t) \neq 0$, find conditions, based on the same Lyapunov function used to establish asymptotic stability, which enable the $\mathcal{L}_2$ gain from the input $w(t)$ to the output $z(t)$ to be bounded as tightly as possible.

Variants of this problem have been studied extensively in the literature and, in particular, Problem 1(1) is, in essence, the absolute stability problem which has been treated since at least the 1940s. Popular solutions to Problem 1(1) are the Circle Criterion and the Popov Criterion. Good introductions to both criteria can be found in [1,2], and comprehensive treatments of the Lyapunov approach found in [3,4]. Further developments of the Popov/Circle Criteria for multiple equilibria are also given [2], although this is beyond the scope of this paper.

Despite the popularity of the Circle and Popov Criteria, it is well known that when more information, other than sector boundedness, is known about the nonlinearity $\Phi(.)$, both criteria can be conservative. In particular, when the nonlinearity is slope-restricted various alternative stability criteria can be derived; there are too many to list here but examples may be found in [5–9]. Of particular note are the results of Zames and Falb [5] which provide a very flexible approach to establishing asymptotic stability for single-input–single-output slope-restricted systems. These results were later extended to various classes of multivariable systems by Safonov and colleagues [10–12]. However, for many years they were not widely used due to the complexity of searching for the so-called Zames–Falb multiplier. Recently several results have become available which, to some extent, automate this search [13–16] and frequently far superior results can be obtained than, for instance, with the Popov Criterion. Despite these improvements, the computational burden associated with, for instance [15,16] tends to be quite high [17,18] due to the search for the multiplier not being "quite" convex (it is an LMI-problem plus a line search). For high-order complex systems, this burden can be prohibitive. In addition, the results provided by such IQC-based methods are not intrinsically associated with the construction of Lyapunov functions, which is in an interesting subject in its own right, and also useful if local results are required.

In [19], a novel Lyapunov function was used in order to obtain less conservative methods for guaranteeing asymptotic stability of the system shown in Fig. 1. The Lyapunov function was piecewise quadratic, as in [20–22], but also used several integral terms derived from the properties of the nonlinearity. Two main LMI-based

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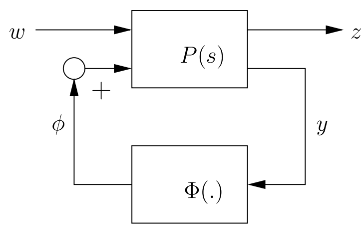

Fig. 1. System under consideration.

results were derived in [19]: Theorem 1 which, although technically correct, featured redundant terms in the LMI's, thereby causing complications and increasing the computational burden; and Theorem 2, which although simpler, seems to feature a small technical error (at least an assumption of controllability on the LTI part appears to be missing) despite "working" in many cases. In addition the $\mathcal{L}_2$ gain problem is not addressed in [19]. However, the work of [19] is significant because it generalises the Popov Criterion and appears dramatically less conservative in many numerical examples, even appearing to out-perform the Zames–Falb multiplier in some cases [17,18].[1] Our goal in this paper is to present results which are an improved alternative to Theorem 2 in [19], that is they are correct, concise and able to provide reasonably non-conservative $\mathcal{L}_2$ gain bounds. The class of system to which the results apply is also extended. The primary motivation for this work is the analysis of the stability of complex systems which can be posed as in Fig. 1, which due to their size/dimension/complexity can be difficult to analyse, without excessive conservatism, using standard results.

The paper is structured as follows: in the next section, the problem is formally introduced. The main results are given in the following section; numerical examples in the section after that. Some brief final remarks conclude the paper.

1.1. Notation

Notation is mainly standard. The $\mathcal{L}_2$ norm of a vector valued function $x(t)$ is defined as $\|x\|_2 := \left(\int_0^\infty \|x(t)\|^2 dt\right)^{1/2}$ where $\|(.)\|$ denotes the standard Euclidean norm; any signal whose $\mathcal{L}_2$ norm is finite is denoted $x(t) \in \mathcal{L}_2$. The nonlinear operator, $\mathcal{T} : w \mapsto z$ is said to have $\mathcal{L}_2$ gain less than γ if $\|z\|_2 < \gamma\|w\|_2 + \beta$ for scalars $\gamma, \beta \geq 0$ and $\forall w \in \mathcal{L}_2$.

2. Problem description

Consider the system depicted in Fig. 1. $P(s)$ denotes a finite-dimensional linear-time-invariant (LTI) system described by the following state-space equations.

$$P(s) \sim \begin{cases} \dot{x} = Ax + B_1 w + B_2\phi \\ z = C_1 x + D_{11} w + D_{12}\phi \\ y = C_2 x + D_{21} w + D_{22}\phi \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^{n_w}$, $z \in \mathbb{R}^{n_z}$, $y \in \mathbb{R}^m$, $\phi = \Phi(y) \in \mathbb{R}^m$ and the state-space matrices are dimensioned accordingly. The nonlinear operator $\Phi(.) : \mathbb{R}^m \mapsto \mathbb{R}^m$ is a decentralised globally Lipschitz, sector bounded, slope restricted nonlinearity which satisfies the following assumptions:

[1] This appears to be more due to the difficulty in the search for Zames–Falb multipliers than something intrinsic however.

Assumption 2. $\Phi(.) : \mathbb{R}^m \mapsto \mathbb{R}^m$ is decentralised, that is for $\sigma \in \mathbb{R}^m$

$$\Phi(\sigma) = \begin{bmatrix}\Phi_1(\sigma_1) & \Phi_2(\sigma_2) & \cdots & \Phi_m(\sigma_m)\end{bmatrix}'$$

and each element, $\Phi_i(.) : \mathbb{R} \mapsto \mathbb{R}$ is globally Lipschitz, zero at zero, and satisfies the following conditions.

$$\frac{\Phi_i(\sigma_i)}{\sigma_i} \in [0, \delta_i] \quad \forall\sigma_i \tag{2}$$

$$\partial\Phi_i(\sigma_i) \in [0, \bar{\delta}_i] \quad \forall\sigma_i \tag{3}$$

for all $i \in \{1, \ldots, m\}$, where $\partial\Phi_i$ represents the sub-differential of Φ_i.

Note that Eq. (3) reflects that fact that $\Phi_i(.)$ may not be differentiable everywhere, but from the Lipschitz assumption, means that

$$\partial\Phi_i(\sigma_i) = \frac{d\Phi_i(\sigma_i)}{d\sigma_i} \quad \text{a.e.} \tag{4}$$

This is a minor technical difference, compared to the original results of [19], which is easily accommodated in the proofs yet is necessary to enable one to treat common slope-restricted nonlinearities such as the saturation and deadzone. In [19] it was shown how the two inequalities (2) and (3) could then be used to derive eight sets of integral inequalities to be used as part of the Lyapunov function. In this work we use only four of those sets of inequalities, but show how the use of such inequalities is able to preserve (and in fact improve upon) Theorem 2 of [19], implying redundancy in those inequalities. For ease of reference we repeat the inequalities used in this paper below, where the $\mu_{I,i}$ are any positive scalars for all $I \in \{1, \ldots, 4\}$, $i \in \{1, \ldots, m\}$.

$$g_{1,i}(x) = \mu_{1,i}\int_0^{y_i} \Phi_i(\sigma_i)d\sigma_i \geq 0 \quad \forall y_i\, \forall i \in \{1, \ldots, m\} \tag{5}$$

$$g_{2,i}(x) = \mu_{2,i}\int_0^{y_i} [\delta_i\sigma_i - \Phi_i(\sigma_i)]d\sigma_i \geq 0 \quad \forall y_i\, \forall i \in \{1, \ldots, m\} \tag{6}$$

$$g_{3,i}(x) = \mu_{3,i}\int_0^{y_i} [\bar{\delta}_i - \partial\Phi_i(\sigma_i)]\sigma_i d\sigma_i \geq 0 \quad \forall y_i\, \forall i \in \{1, \ldots, m\} \tag{7}$$

$$g_{4,i}(x) = \mu_{4,i}\int_0^{y_i} \partial\Phi_i(\sigma_i)[\delta_i\sigma_i - \Phi_i(\sigma_i)]d\sigma_i \geq 0 \quad \forall y_i\, \forall i \in \{1, \ldots, m\}. \tag{8}$$

In addition, from Eq. (2) with $\sigma(t) = y(t)$, the standard sector inequality follows:

$$\mathcal{S}_\Delta = 2\phi'\mathbf{N_1}(\Delta y - \phi) \geq 0 \tag{9}$$

$$= 2\phi'\mathbf{N_1}(\Delta(C_2 x + D_{21}w + D_{22}\phi) - \phi) \geq 0 \tag{10}$$

where $\Delta := \mathrm{diag}(\delta_1, \ldots, \delta_m) > 0$ and $\mathbf{N_1} > 0$ is any positive definite diagonal matrix. Also, from Eq. (3), at the values of y at which $\Phi(y)$ is differentiable, we have

$$\dot{\phi}_i(\bar{\delta}_i\dot{y}_i - \dot{\phi}_i) \geq 0. \tag{11}$$

Thus, as $\Phi(.)$ is Lipschitz, we have the following inequality

$$\mathcal{S}_{\bar{\Delta}} = 2\dot{\phi}'\mathbf{N_2}(\bar{\Delta}\dot{y} - \dot{\phi}) \geq 0 \quad \text{a.e.} \tag{12}$$

where $\bar{\Delta} := \mathrm{diag}(\bar{\delta}_1, \ldots, \bar{\delta}_m) > 0$ and $\mathbf{N_2} > 0$ is any positive definite diagonal matrix. Evaluation of the above expression requires knowledge of $\dot{y}$, which in turn requires knowledge of $\dot{w}$, which is