Systems & Control Letters 69 (2014) 7–15

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

The optimal control related to Riemannian manifolds and the viscosity solutions to Hamilton–Jacobi–Bellman equations[☆]

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ARTICLE INFO

Article history:
Received 26 June 2010
Received in revised form
11 July 2013
Accepted 31 March 2014
Available online 4 May 2014

Keywords:
Riemannian manifold
Hamilton–Jacobi–Bellman equation
Backward stochastic differential equations
Dynamic programming principle
Viscosity solution

ABSTRACT

In this paper we study the optimal stochastic control problem for stochastic differential equations on Riemannian manifolds. The cost functional is specified by controlled backward stochastic differential equations in Euclidean space. Under some suitable assumptions, we conclude that the value function is the unique viscosity solution to the associated Hamilton–Jacobi–Bellman equation which is a fully nonlinear parabolic partial differential equation on Riemannian manifolds.

1. Introduction

General nonlinear backward stochastic differential equations (BSDEs) were introduced independently by Pardoux and Peng [1] in 1990 and Duffie and Epstein [2] in 1992. Since then this theory has been widely applied in stochastic control and game theory, mathematical finance, partial differential equations (PDEs), nonlinear expectations after which were established.

Among these applications, the probabilistic interpretation for PDEs is very interesting and popular. As we know, a solution of a linear second order parabolic (or elliptic) equation can be formulated as a functional of a solution of some stochastic differential equation (SDE) (see [3] for detailed references). For nonlinear case, Peng [4] gave a probabilistic interpretation for systems of quasilinear parabolic PDEs. In 1992, Peng [5] used stochastic optimal control theory to obtain the probabilistic interpretation for one kind of fully nonlinear second-order PDE which is the well-known Hamilton–Jacobi–Bellman (HJB) equation. For more details in this field, the reader is referred to El Karoui, Peng and Quenez [6], Peng [7], Barles, Buckdahn and Pardoux [8], Peng and Wu [9], Wu and Yu [10], Buckdahn and Li [11], etc. Thus the theory of the probabilistic interpretation for PDEs provides a powerful tool for studying wider classes of nonlinear equations of parabolic and elliptic types.

Since there are many important nonlinear equations which are full of geometrical meaning, a natural problem is: can one obtain a similar interpretation for a system of nonlinear PDEs on Riemannian manifolds? This is the objective of this paper. More recently there appear some research papers on the theory of viscosity solutions to general second order PDEs on Riemannian manifolds which makes the probabilistic interpretation for nonlinear PDEs on Riemannian manifolds be possible.

Since Crandall and Lions [12] introduced the notion of viscosity solutions to nonlinear PDEs on R^n in 1980s, this theory has been applied widely and was enriched and expanded by many mathematicians. We can refer the reader to [12] and the references given therein. There have been various approaches to extend the theory of viscosity solutions of first order Hamilton–Jacobi equations, and the corresponding nonsmooth calculus, to the setting of Riemannian manifolds, see Mantegazza and Mennucci [13], Azagra, Ferrera and López-Mesas [14], Ledyaev and Zhu [15] and Gursky and Viaclovsky [16]. For the case of general second order PDEs, the reader is referred to Azagra, Ferrera and Sanz [17], Azagra, Jiménez-sevilla and Macià [18], Peng and Zhou [19] and Zhu [20].

In this paper, we investigate the optimal stochastic control SDEs on Riemannian manifolds. The cost functional is specified by controlled BSDEs in Euclidean space. The related HJB equation is a fully nonlinear second order PDE on Riemannian manifolds. To be more detail, we aim to give a probabilistic interpretation for the solution of the following HJB equation:

$$\begin{cases}\partial_t u(t,x) + H(t,x,u,du,d^2u) = 0 & \text{in } [0,T)\times M,\\ u(T,x) = \Phi(x), \quad x\in M,\end{cases} \tag{1.1}$$

[☆] This work is supported by the National Natural Science Foundation of China (11101209) and partially supported by the National Natural Science Foundation of China (11201262) and the National Natural Science Foundation of China (11301274).

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http://dx.doi.org/10.1016/j.sysconle.2014.03.011

where du, d^2u mean $d_x u(t,x)$ and $d_x^2 u(t,x)$ and the Hamiltonian

$H : [0,T] \times M \times R \times TM_x^* \times \mathcal{L}_s^2(TM_x) \to R$

is defined as follows:

$$H(t,x,r,\zeta,A) = \inf_{v\in U}\left\{ \frac{1}{2}\sum_{\alpha=1}^{d} v_\alpha^2 \langle AV_\alpha(t,x), V_\alpha(t,x)\rangle + \langle \zeta, v_0 V_0(t,x)\rangle + f(t,x,r,\{\langle \zeta, v_\alpha V_\alpha(t,x)\rangle\}_{\alpha=1}^d, v)\right\}.$$

Here, M is a compact Riemannian manifold without boundary, TM_x^* stands for the cotangent space of M at a point x, TM_x stands for the tangent space at x and $\mathcal{L}_s^2(TM_x)$ denotes the symmetric bilinear forms on TM_x. U is a compact subset of R^{d+1} and $V_0, V_1, \ldots, V_d$ are $d+1$ deterministic one-parameter smooth vector fields on M. The functions f and Φ are supposed to satisfy (A1) and (A2), more details in Sections 2 and 4. For this, we consider the following controlled SDE on M in a fixed time interval $[t,T]$:

$$\begin{cases} dX_s^{t,\zeta;v.} = v_0(s)V_0(s, X_s^{t,\zeta;v.})ds + \sum_{\alpha=1}^{d} V_\alpha(s, X_s^{t,\zeta;v.}) \circ v_\alpha(s) dW_s^\alpha, \\ X_t^{t,\zeta;v.} = \zeta \in M, \end{cases}$$

and the associated real-valued BSDE is as follows:

$$\begin{cases} -dY_s^{t,\zeta;v.} = f(s, X_s^{t,\zeta;v.}, Y_s^{t,\zeta;v.}, Z_s^{t,\zeta;v.}, v_s) - Z_s^{t,\zeta;v.} dW_s, \\ \quad s \in [t,T], \\ Y_T^{t,\zeta;v.} = \Phi(X_T^{t,\zeta;v.}). \end{cases}$$

Under assumptions (A1) and (A2), they have unique solutions $X_\cdot^{t,\zeta;v.}$ and $(Y_\cdot^{t,\zeta;v.}, Z_\cdot^{t,\zeta;v.})$, respectively. When $\zeta = x \in M$ is deterministic, for any admissible control $v(\cdot)$, the cost functional is defined by

$$J(t,x;v(\cdot)) := Y_s^{t,x;v.}|_{s=t} = Y_t^{t,x;v.}, \quad (t,x) \in [0,T] \times M.$$

The value function of our stochastic optimal control problem is:

$$u(t,x) := \operatorname{essinf}_{v(\cdot)\in\mathcal{U}_{t,T}} J(t,x;v(\cdot)), \quad (t,x) \in [0,T] \times M,$$

where $\mathcal{U}_{t,T}$ denotes the set of all admissible controls.

The BSDE method developed by Peng [5,7] for the dynamic programming of stochastic optimal control is extended into this paper. We will prove that the value function $u(t,x)$ is continuous in $(t,x) \in [0,T] \times M$, satisfies the dynamic programming principle (DPP) and is the unique viscosity solution of the associated HJB equation (1.1). However, the proofs become more technical. The square of the distance function on a manifold is not necessarily twice differentiable and we cannot apply Itô's formula directly. Therefore we lose many classical estimates about the solutions of SDEs and BSDEs and must turn to the embedding mapping.

The paper is organized as follows: In Section 2, we introduce the framework of the stochastic optimal control problem. In Section 3, we prove that $u(t,x)$ is continuous in $(t,x) \in [0,T] \times M$ and satisfies the DPP. We show in Section 4 that the value function $u(t,x)$ is the unique viscosity solution of the associated HJB equation (1.1) which implies a new existence result for a viscosity solution.

2. Framework

Let $(W(t), t \geq 0)$ be a d-dimensional standard Brownian motion on some complete probability space $(\Omega, \mathcal{F}, P)$. We denote by $(\mathcal{F}_t)_{t\geq 0}$ the natural filtration generated by W and augmented by the P-null sets of $\mathcal{F}$.

Let U be a compact subset of R^{d+1}. We call a function $h : \Omega \times [t,T] \to U$ an admissible control if it is an adapted stochastic process. We denote by $\mathcal{U}_{t,T}$ the set of all admissible controls.

Assume that M is a compact Riemannian manifold without boundary. Let us consider the following controlled SDE on M in a fixed time interval $[t,T]$:

$$\begin{cases} dX_s^{t,\zeta;v.} = v_0(s)V_0(s, X_s^{t,\zeta;v.})ds \\ \quad + \sum_{\alpha=1}^{d} V_\alpha(s, X_s^{t,\zeta;v.}) \circ v_\alpha(s) dW_s^\alpha, \\ X_t^{t,\zeta;v.} = \zeta \in M, \end{cases} \tag{2.1}$$

where ζ is $\mathcal{F}_t$-measurable, $v. = v(\cdot) := (v_0(\cdot), v_1(\cdot), \ldots, v_d(\cdot)) \in \mathcal{U}_{t,T}$, and $V_0, V_1, \ldots, V_d$ are $d+1$ deterministic one-parameter smooth vector fields on M.

Since M is compact and without boundary, according to [21], there exists a unique M-valued continuous process which solves Eq. (2.1). Moreover, this solution does not explode.

Let us consider functions $f : [0,T] \times M \times R \times R^d \times U \to R$ and $\Phi : M \to R$ which satisfy:

(A1) There exists a constant $K \geq 0$, s.t., for all $t \in [0,T]$, $x, x' \in M$, $y, y' \in R$, $z, z' \in R^d$, $v, v' \in U$,

$$|\Phi(x) - \Phi(x')| + |f(t,x,y,z,v) - f(t,x',y',z',v')| \leq K(|y-y'| + |z-z'| + d(x,x') + |v-v'|).$$

(A2) There exists a constant $K_0 \geq 0$, s.t., for all $t \in [0,T]$, $x \in M$, $v \in U$,

$$|f(t,x,0,0,v)| \leq K_0,$$

where $d(\cdot,\cdot)$ denotes the Riemannian distance function on M.

Under the above assumptions, according to [7], there exists a unique pair $(Y_\cdot^{t,\zeta;v.}, Z_\cdot^{t,\zeta;v.}) \in \mathcal{M}(t,T;R \times R^d)$ which solves the following BSDE:

$$\begin{cases} -dY_s^{t,\zeta;v.} = f(s, X_s^{t,\zeta;v.}, Y_s^{t,\zeta;v.}, Z_s^{t,\zeta;v.}, v_s) - Z_s^{t,\zeta;v.} dW_s, \\ \quad s \in [t,T], \\ Y_T^{t,\zeta;v.} = \Phi(X_T^{t,\zeta;v.}), \end{cases} \tag{2.2}$$

where $\mathcal{M}(0,T;R^n)$ denotes the Hilbert space of adapted stochastic processes $\phi : \Omega \times [0,T] \to R^n$ such that

$$\|\phi\| = \left(E\int_0^T |\phi(t)|^2 dt\right)^{\frac{1}{2}} < \infty.$$

When $\zeta = x \in M$ is deterministic, we define

$$J(t,x;v(\cdot)) := Y_s^{t,x;v.}|_{s=t} = Y_t^{t,x;v.}, \quad (t,x) \in [0,T] \times M.$$

This is the so-called cost functional. And then we can define the value function of the optimal control problem as follows:

$$u(t,x) := \operatorname{essinf}_{v(\cdot)\in\mathcal{U}_{t,T}} J(t,x;v(\cdot)), \quad (t,x) \in [0,T] \times M. \tag{2.3}$$

Our purpose is to get the general DPP and give the probabilistic interpretation for the associated HJB equation (1.1).

3. Dynamic programming principle

Define

$$\mathcal{U}_{t,T}^t := \{v(\cdot) \in \mathcal{U}_{t,T} : v(\cdot) \text{ is } \mathcal{F}_s^t\text{-adapted}\},$$

where $\mathcal{F}_s^t := \sigma\{W_r - W_t, t \leq r \leq s\}$.

By Proposition 5.1 in [7], there exist $\{v^i(\cdot)\}_{i=1}^\infty$, $v^i(\cdot) \in \mathcal{U}_{t,T}^t$, such that

$$u(t,x) = \lim_{i\to\infty} J(t,x;v^i(\cdot))$$

and $u(t,x)$ is a deterministic function, i.e.,

$$u(t,x) := \operatorname{essinf}_{v(\cdot)\in\mathcal{U}_{t,T}} J(t,x;v(\cdot)) = \inf_{v(\cdot)\in\mathcal{U}_{t,T}^t} J(t,x;v(\cdot)).$$