Systems & Control Letters 69 (2014) 38–46

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Approximately bisimilar symbolic models for randomly switched stochastic systems ☆

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ARTICLE INFO

Article history:
Received 3 January 2014
Received in revised form
3 April 2014
Accepted 14 April 2014
Available online 8 May 2014

Keywords:
Stochastic hybrid systems
Randomly switched models
Symbolic models
Finite abstractions
Formal synthesis

ABSTRACT

In the past few years there has been a growing interest in the use of symbolic models for control systems. The main reason is the possibility to leverage algorithmic techniques over symbolic models to synthesize controllers that are valid for the concrete control systems. Such controllers can enforce complex logical specifications that are otherwise hard (if not impossible) to establish on the concrete models with classical control techniques. Examples of such specifications include those expressible via linear temporal logic or as automata on infinite strings. A relevant goal in this research line is in the identification of classes of systems that admit symbolic models: in particular, continuous-time systems with stochastic or hybrid dynamics have been only recently considered, due to their rather general and complex dynamics. In this work we make progress in this direction by enlarging the class of stochastic hybrid systems admitting finite, symbolic models: specifically, we show that randomly switched stochastic systems, satisfying some incremental stability assumption, admit such models.

1. Introduction

Stochastic hybrid systems represent a general class of dynamical systems that combine continuous dynamics with discrete components and that are affected by continuous probabilistic terms as well as discrete random events. Numerous real-life systems from fields such as biochemistry [1], air traffic control [2], systems biology [3], and communication networks [4], can be modeled as stochastic hybrid systems. Randomly switched stochastic systems, also known as *switching* stochastic systems [5], are a relevant subclass of general stochastic hybrid systems. They consist of a finite family of subsystems (modes, or locations), together with a random *switching signal* that specifies the active subsystem at every time instant. Each subsystem is further endowed with continuous probabilistic dynamics, described by a control-dependent stochastic differential equation.

Quite some research has recently focused on characterizing classes of systems, involving continuous and possibly discrete components, that admit symbolic models. A symbolic model is a finite discrete approximation of a concrete model, resulting from replacing equivalent (sets of) continuous states by discrete symbols. Symbolic models are interesting because they allow the application of algorithmic machinery for controller synthesis on discrete systems [6] towards the synthesis of hybrid controllers for the corresponding concrete complex models. Such controllers are synthesized to satisfy classes of specifications that traditionally have not been considered in the context of control theory: these include specifications involving regular languages and temporal logics [7].

The search for classes of continuous-time stochastic systems admitting symbolic models include results on stochastic dynamical systems under contractivity assumptions [8], which are valid only for autonomous models (i.e. with no control input); on probabilistic rectangular automata [9] endowed with random behaviors exclusively on their discrete components and with simple continuous dynamics; on linear stochastic control systems [10], however without any quantitative relationship between abstract and concrete models; on stochastic control systems without any stability assumptions, but with no hybrid dynamics [11]; on incrementally-stable stochastic control systems without discrete components [12] and without requiring state-space discretization [13]; and finally on incrementally-stable stochastic switched

☆ This work is supported by the European Commission STREP project MoVeS 257005.

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http://dx.doi.org/10.1016/j.sysconle.2014.04.003
0167-6911/

systems [14] where the discrete dynamics, in the form of mode changes, are governed by a non-probabilistic control signal. The results in [11–14] are based on the notion of (alternating) approximate (bi)simulation relation, introduced in [15,16]. Notions of bisimulation for continuous-time stochastic hybrid systems have also been studied in [17], although with a different goal than that of synthesizing symbolic models: while we are interested in the construction of bisimilar models that are finite, the work in [17] uses bisimulation to relate continuous (and thus infinite) stochastic hybrid systems. Finally, there exist discretization results based on weak approximations of continuous-time stochastic control systems [18] and of continuous-time stochastic hybrid systems [19], however these do not provide any explicit approximation bound.

To the best of our knowledge there is no work on the construction of finite bisimilar abstractions for continuous-time switching stochastic systems where the discrete dynamics, in the form of mode changes, are governed by a random switching signal. Models for these systems have become ubiquitous in engineering applications, such as power electronics [20], manufacturing [21], economic and finance [22]: automated controller synthesis techniques for this class of models can thus lead to more reliable system development at lower costs and times.

The main contribution of this paper is to show that switching stochastic systems, under some incremental stability assumption, admit symbolic models that are alternatingly approximately bisimilar to the concrete ones, with a precision (say ε) that can be chosen a-priori, as a design parameter. More precisely, by guaranteeing the existence of an alternating ε-approximate bisimulation relation between concrete and symbolic models, one deduces that there exists a controller enforcing a desired complex specification on the symbolic model if and only if there exists a hybrid controller enforcing an ε-specification on the original switching stochastic system. We show the description of the discussed incremental stability property in terms of a so-called common Lyapunov function (with requires no probabilistic structure on the switching signal), or alternatively in terms of multiple Lyapunov functions with some fairly general probabilistic structure on the switching signal.

Building upon [12,14], the result of this paper extends that in [12] from a single stochastic control system to a number of randomly switching stochastic systems, and the result in [14] from multiple stochastic dynamical systems with mode changes that are governed by a non-probabilistic controlled signal to multiple stochastic control systems in which mode changes are governed by a random (uncontrolled) signal. The presence of a randomly switching signal in this paper requires to provide novel symbolic models: these allow transferring the synthesized control strategies directly to the original system, regardless of the particular evolution of the switching signal.

2. Randomly switched stochastic systems

2.1. Notation

The identity map on a set A is denoted by 1_A. If A is a subset of B, we denote by $\imath_A : A \hookrightarrow B$ or simply by $\imath$ the natural inclusion map taking any $a \in A$ to $\imath(a) = a \in B$. Given a set $A \subseteq \mathbb{R}^n$, the symbol $\overline{A}$ denotes the topological closure of A. The symbols $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{R}_0^+$ denote the set of natural, nonnegative integer, integer, real, positive, and nonnegative real numbers, respectively. The symbols 0_n and $0_{n\times m}$ denote the zero vector and matrix in $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively. Given a vector $x \in \mathbb{R}^n$, we denote by x_i the ith element of x, and by $\|x\|$ the infinity norm of x, namely, $\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$, where $|x_i|$ denotes the absolute value of x_i. Given matrices $M = \{m_{ij}\} \in \mathbb{R}^{n\times m}$ and $P = \{p_{ij}\} \in \mathbb{R}^{n\times n}$, we denote by $\|M\|$ the infinity norm of M, namely, $\|M\| = \max_{1\leq i\leq n}\sum_{j=1}^{m}|m_{ij}|$; by $\text{Tr}(P)$ the trace of P, namely, $\text{Tr}(P) = \sum_{i=1}^{n} p_{ii}$; by $\|M\|_F$ the Frobenius norm of M, namely, $\|M\|_F = \sqrt{\text{Tr}\left(MM^T\right)}$; and by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ the minimum and maximum eigenvalues of a symmetric matrix P, respectively. We denote by Δ the *diagonal set*, namely, $\Delta = \{(x, x) \mid x \in \mathbb{R}^n\}$.

The closed ball centered at $x \in \mathbb{R}^n$ with radius λ is defined by $\mathcal{B}_\lambda(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq \lambda\}$. A set $B \subseteq \mathbb{R}^n$ is called a *box* if $B = \prod_{i=1}^{n}[c_i, d_i]$, where $c_i, d_i \in \mathbb{R}$ with $c_i < d_i$ for each $i \in \{1, \ldots, n\}$. The *span* of a box B is defined as $span(B) = \min\{|d_i - c_i| \mid i = 1, \ldots, n\}$. By defining $[\mathbb{R}^n]_\eta = \{a \in \mathbb{R}^n \mid a_i = k_i\eta, k_i \in \mathbb{Z}, \ i = 1, \ldots, n\}$, the set $\bigcup_{p\in[\mathbb{R}^n]_\eta} \mathcal{B}_\lambda(p)$ is a countable covering of $\mathbb{R}^n$ for any $\eta \in \mathbb{R}^+$ and $\lambda \geq \eta/2$. For a box B and $\eta \leq span(B)$, define the η-approximation $[B]_\eta = [\mathbb{R}^n]_\eta \cap B$. Note that $[B]_\eta \neq \varnothing$ for any $\eta \leq span(B)$ and that for any $\eta \in \mathbb{R}^+$ with $\eta \leq span(B)$ and $\lambda \geq \eta$, we have $B \subseteq \bigcup_{p\in[B]_\eta} \mathcal{B}_\lambda(p)$. We extend the notions of span and of approximation to finite unions of boxes as follows. Let $A = \bigcup_{j=1}^{M} A_j$, where each A_j is a box. Define $span(A) = \min\{span(A_j) \mid j = 1, \ldots, M\}$, and for any $\eta \leq span(A)$, define $[A]_\eta = \bigcup_{j=1}^{M}[A_j]_\eta$.

Given a set X and a metric $\mathbf{d} : X \times X \to \mathbb{R}_0^+$, we denote by $\mathbf{d}_h$ the Hausdorff pseudometric induced by $\mathbf{d}$ on 2^X; we recall that for any $X_1, X_2 \subseteq X$, $\mathbf{d}_h(X_1, X_2) := \max\left\{\vec{\mathbf{d}}_h(X_1, X_2), \vec{\mathbf{d}}_h(X_2, X_1)\right\}$, where $\vec{\mathbf{d}}_h(X_1, X_2) = \sup_{x_1\in X_1}\inf_{x_2\in X_2}\mathbf{d}(x_1, x_2)$ is the directed Hausdorff pseudometric. Given a measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}^n$, the (essential) supremum (sup norm) of f is denoted by $\|f\|_\infty$; we recall that $\|f\|_\infty = (\text{ess})\sup\{\|f(t)\|, \ t \geq 0\}$. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if $\gamma \in \mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed nonzero r, the map $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. We identify a relation $R \subseteq A \times B$ with the map $R : A \to 2^B$ defined by $b \in R(a)$ iff $(a, b) \in R$. Given a relation $R \subseteq A \times B$, R^{-1} denotes the inverse relation defined by $R^{-1} = \{(b, a) \in B \times A : (a, b) \in R\}$.

2.2. Randomly switched (a.k.a. switching) stochastic systems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions of completeness and right-continuity [23, p. 48]. Let $\{W_t\}_{t\geq 0}$ be a $\widehat{q}$-dimensional $\mathbb{F}$-adapted Brownian motion [24].

Definition 2.1. A switching stochastic system is a tuple $\Sigma = (\mathbb{R}^n, \mathsf{U}, \mathcal{U}, \mathsf{P}, \mathcal{P}, F, G)$, where

- $\mathbb{R}^n$ is the continuous state space;
- $\mathsf{U} \subseteq \mathbb{R}^m$ is a compact input set;
- $\mathcal{U}$ is a subset of the set of all measurable functions of time, from $\mathbb{R}_0^+$ to U;
- $\mathsf{P} = \{1, \ldots, m\}$ is a finite set of modes;
- $\mathcal{P}$ is a subset of the set of all piecewise constant càdlàg (i.e. right-continuous and with left limits) functions of time from $\mathbb{R}_0^+$ to P, and characterized by a finite number of discontinuities on every bounded interval in $\mathbb{R}_0^+$ (no Zeno behavior);
- $F = \{f_1, \ldots, f_m\}$ is such that, for any $p \in \mathsf{P}$, $f_p : \mathbb{R}^n \times \mathsf{U} \to \mathbb{R}^n$ satisfies the following Lipschitz assumption: there exist constants $L_x^p, L_u^p \in \mathbb{R}^+$ such that $\|f_p(x, u) - f_p(x', u')\| \leq L_x^p\|x - x'\| + L_u^p\|u - u'\|$, for all $x, x' \in \mathbb{R}^n$ and all $u, u' \in \mathsf{U}$;
- $G = \{g_1, \ldots, g_m\}$ is such that, for any $p \in \mathsf{P}$, $g_p : \mathbb{R}^n \to \mathbb{R}^{n\times\widehat{q}}$ satisfies the following Lipschitz assumption: there exists a constant $Z_p \in \mathbb{R}^+$ such that, for all $x, x' \in \mathbb{R}^n$: $\|g_p(x) - g_p(x')\| \leq Z_p\|x - x'\|$.