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Exponential stabilization of linear systems with time-varying delayed state feedback via partial spectrum assignment

F. Cacace [a,*], A. Germani [a,b], C. Manes [b]

[a] Università Campus Bio-Medico, Via Àlvaro del Portillo, 21, 00128 Roma, Italy

[b] Dip. di Ingegneria e Scienze dell'Informazione e Matematica, Università degli Studi dell'Aquila, Via Vetoio, 67100 Coppito (AQ), Italy

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ABSTRACT

We consider the problem of controlling a linear system when the state is available with a known time-varying delay (delayed-state feedback control) or the actuator is affected by a delay. The solution proposed in this paper consists in partially assigning the spectrum of the closed-loop system to guarantee the exponential zero-state stability with a prescribed decay rate by means of a finite-dimensional control law. A non conservative bound on the maximum allowed delay for the prescribed decay rate is presented, which holds for both cases of constant and time-varying delays. An advantage over recent and similar approaches is that differentiability or continuity of the delay function is not required. We compare the performance of our approach, in terms of delay bound and input signal, with another recent approach.

1. Introduction

Since the birth of automatic control the problem of stabilizing systems with delays in the state and/or input variables has been investigated by a multitude of researchers (see *e.g.* [1–6] and the references therein). For the general case of linear systems with delays in the state equations several control approaches have been developed, including Finite Spectrum Assignment [7,8], Continuous Pole Placement [9], optimization based pole assignment [10], Infinite Spectrum Assignment [11], Lyapunov–Krasovskii functional based methods [12,13], parametric Lyapunov equations [14], and Matrix Lambert Function [15,16]. Predictor-based approaches for controlling time-delay systems have long been used for systems with delayed input, and have received renewed interest in recent years, even if most approaches only consider the case of constant delay [17,18]. In [19] a sequence of predictors is used to overcome the delay limitation in the constant delay case. Predictor-based state feedback and output feedback controllers have been proposed in [20,21] for nonlinear systems with time-varying input delays. State predictors for nonlinear systems with output delays have been studied in [22,23].

Traditional predictor-based controllers use infinite-dimensional feedback laws, whose accurate implementation has been widely discussed in the literature (see for example [24,5,25]). A feedback law that only involves finite dimensional static state feedback is simpler to implement, because it does not require to store the previous values of the state and it does not need discretization. An approach of this kind, named Truncated Prediction Feedback, has been used in [26] for linear systems with constant delays that are not exponentially unstable and extended in [27] to the case of time-varying input delays and in [28] to exponentially unstable systems.

In this paper we use a finite-dimensional method for linear systems with no restriction on the position of the poles of the open-loop system, when the state is available with a possibly time-varying delay. This is equivalent to solving a delayed input stabilization problem when the delay function is known in advance (see [27] for details). It can be seen as an extension to the variable delay case of the general structure of predictors outlined in [29], and it improves the recent results cited above in several regard. In particular, continuity and differentiability of the delay function are not needed, and the delay bound for a prescribed exponential rate of the controlled system is the same in the constant and variable delay case. Sufficient and, under some assumptions, necessary conditions on the delay bound are easy to check. The work here presented extends the preliminary results in [30], where only the case of single-input systems has been investigated.

This paper is organized as follows. In Section 2 the control problem is formally stated, and preliminary definitions are given. In Section 3 the proposed control law and its features are presented. Section 4 investigates stability conditions for time-varying delays.

* Corresponding author. Tel.: +39 3496570566.
E-mail addresses: f.cacace@unicampus.it, filippo.cacace@gmail.com (F. Cacace), alfredo.germani@univaq.it (A. Germani), costanzo.manes@univaq.it (C. Manes).

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We compare the features of the proposed method with the recent approach presented in [28] in Section 5, and a numerical comparison is provided in Section 6.

Notation. Given a real number α, the symbol $\mathbb{C}_{>\alpha}$ ($\mathbb{C}_{\geq\alpha}$) denotes the set of all complex numbers s such that $\Re(s) > \alpha$ ($\Re(s) \geq \alpha$). $S_{\mathbb{C}}$ denotes the set of all the countable subsets of $\mathbb{C}$ that are symmetric w.r.t. the real axis (i.e., if $U \in S_{\mathbb{C}}$, then $z \in U \Rightarrow z^* \in U$). $\sigma(A) \in S_{\mathbb{C}}$ denotes the spectrum of a real square matrix A. Given a positive real number $\bar{\delta}$ and an integer n, the symbol $\mathcal{C}^n_{\bar{\delta}}$ denotes the space of continuous functions that map $[-\bar{\delta}, 0]$ in $\mathbb{R}^n$, with the uniform convergence norm, denoted $\|\cdot\|_\infty$. For a set $L \in S_{\mathbb{C}}$, $\mu(L) = \max_{\lambda_i \in L}\{\Re(\lambda_i)\}$. Abbreviations: LTI: Linear Time Invariant; LTDS: Linear Time Delay System; DDE: delay-differential equation.

2. Problem statement

Consider a linear system whose state is available with a known time-delay δ_t, possibly time-varying in a given interval $[0, \bar{\delta}]$:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad t \geq 0, \\ \xi(t) &= x(t-\delta_t), \quad \delta_t : \mathbb{R}_+ \mapsto [0, \bar{\delta}], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^p$ is the input. The pair $(\xi(t), \delta_t)$ is the measurement available at time t. Under the assumption that the pair (A, B) is controllable, we consider the problem of constructing a stabilizing feedback control law with delay-dependent gain matrix:

$$u(t) = -K(\delta_t)\,\xi(t), \qquad K : [0, \bar{\delta}] \mapsto \mathbb{R}^{p\times n}. \tag{2}$$

We assume that the control law (2) starts operating at time $t = 0$. Thus, the closed-loop system is a time-delay system with the following structure:

$$\begin{aligned} \dot{x}(t) &= Ax(t) - BK(\delta_t)\,x\big(t-\delta_t\big), \quad t \geq 0, \\ x(t) &= \phi(t), \quad t \in [-\bar{\delta}, 0], \delta_t : \mathbb{R}_+ \mapsto [0, \bar{\delta}], \end{aligned} \tag{3}$$

where $\phi \in \mathcal{C}^n_{\bar{\delta}}$ is the so called *preshape function*. Of course, the gain function $K(\cdot)$ and the delay function δ_t must be such to ensure that the solution of (3) exists and is unique. For this reason we assume $K(\cdot)$ continuous in $[0, \bar{\delta}]$ and δ_t measurable in $\mathbb{R}_+$. Note that the control problem described above is equivalent to the problem of controlling a linear system when the state is available without delay and the actuator is affected by a known time-varying delay.

We are interested in designing control laws that ensure exponential decay of the state with a prescribed rate. The following definition is useful for our purposes.

Definition 1 (*α-exp Stability*)**.** For a given real number $\alpha > 0$, the system (3) is said to be *α-exp stable* if there exists $\gamma > 0$ such that

$$\|x(t)\| \leq e^{-\alpha t}\gamma\,\|\phi\|_\infty, \quad \forall t \geq 0, \forall \phi \in \mathcal{C}^n_{\bar{\delta}}. \tag{4}$$

Definition 2. Consider the system (1) with a given control law of the type (2). For a given $\alpha > 0$ the *maximal delay for α-exp stability*, denoted Δ_α, is the supremum among all $\bar{\delta} > 0$ such that the closed loop system (3) is α-exp stable for any measurable $\delta_t \in [0, \bar{\delta}]$. If the system (3) is α-exp stable for any $\delta_t \in [0, \infty)$, then $\Delta_\alpha = \infty$. Δ_0 denotes the *maximal delay* for asymptotic stability.

Definition 2 of Δ_α implies that if $\bar{\delta} > \Delta_\alpha$, then there exists at least one delay function $\delta_t \in [0, \bar{\delta}]$ such that the system (3) is not α-exp stable.

Note that if the delay-dependent gain $K(\delta_t)$ is properly chosen the system (3), although time-varying, may admit solutions of the type $e^{\lambda t}v$, for some *constant* $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$. Thus, we can give the following:

Definition 3. A *normal mode* of the LTDS (3), where the gain function $K(\cdot)$ and the delay function $\delta_t \in [0, \bar{\delta}]$ are given, is a solution of (3) of the type $x(t) = e^{\lambda t}v$, with $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n$. If such a solution exists, λ is called a *modal number* of (3). $\mathcal{M}\big(A, -BK(\delta_t)\big) \subset \mathbb{C}$ will denote the *set of modal numbers* of the system (3).

Of course, the *set of modal numbers* $\mathcal{M}\big(A, -BK(\delta_t)\big)$ extends the concept of spectrum to time-varying systems, and it can be an empty set if $K(\cdot)$ is not properly chosen. Like the spectrum, if A, B and $K(\cdot)$ are real and λ is a modal number, then also λ^* is a modal number, i.e. $\mathcal{M}\big(A, -BK(\delta_t)\big) \in S_{\mathbb{C}}$. For $\delta_t = 0$ we have $\mathcal{M}\big(A, -BK(0)\big) = \sigma\big(A - BK(0)\big)$, and for constant delay $\delta_t = \delta > 0$, the set $\mathcal{M}\big(A, -BK(\delta)\big)$ is made of the countably infinite roots of the characteristic function

$$\nu_\delta(s) = \left|sI_n - A + BK(\delta)\,e^{-s\delta}\right|, \tag{5}$$

i.e. $\lambda \in \mathcal{M}\big(A, -BK(\delta)\big) \iff \nu_\delta(\lambda) = 0$. Let us define $\bar{\sigma}_\delta = \max\{\Re(\lambda), \lambda \in \mathcal{M}\big(A, -BK(\delta)\big)\}$. It is known that $\bar{\sigma}_\delta$ exists and $\bar{\sigma}_\delta < \infty$ (Lemma 1.4.1 in [31]). Moreover, if $\bar{\sigma}_\delta < 0$, then the system (3) is α-exp stable with $\alpha \in (0, |\bar{\sigma}_\delta|)$ (Thm. 1.6.2 in [31]). These results can be summarized in the following Proposition:

Proposition 1. *Consider the system (3), with a constant delay $\delta_t = \delta$. For a given real number $\alpha > 0$ the system is α-exp stable if $\nu_\delta(s) \neq 0$ $\forall s \in \mathbb{C}_{\geq -\alpha}$, and only if $\nu_\delta(s) \neq 0$ $\forall s \in \mathbb{C}_{>-\alpha}$.*

Control problem formulation: Partial Spectrum Assignment with α-exp stability.

Let $L = \{\lambda_1, \ldots, \lambda_n\} \in S_{\mathbb{C}}$ denote a set of n real or complex numbers. The problem of Partial Spectrum Assignment (PSA) consists in finding a time-dependent feedback gain $K : [0, \bar{\delta}] \mapsto \mathbb{R}^{p\times n}$, such that with the feedback law (2) the set L is included in the set of modal numbers for any delay function $\delta_t \in [0, \bar{\delta}]$, i.e. $L \subset \mathcal{M}\big(A, -BK(\delta_t)\big), \forall \delta_t \in [0, \bar{\delta}]$. The PSA problem with α-exp stability (PSA$^\alpha$) requires that, in addition, the system (3) is α-exp stable.

Of course, a necessary condition for having α-exp stability is that $\mu(L) \leq -\alpha$. Note that when $\delta_t = 0$, if the pair (A, B) is controllable, then for any choice of $L \in S_{\mathbb{C}}$, there exists $\overline{K} \in \mathbb{R}^{p\times n}$ such that $\sigma(A - B\overline{K}) = L$.

3. The feedback law

This section summarizes the main results of the paper. For the sake of clarity, their formal statement and corresponding proof is postponed to Section 4 or to Appendices A and B as detailed in the sequel.

Given a set $L \in S_{\mathbb{C}}$ of n desired eigenvalues for the system (1), where the pair (A, B) is assumed controllable, consider the following delay-dependent feedback law

$$u(t) = -K(\delta_t)\xi(t), \quad \text{with } K(\delta_t) = \overline{K}e^{\overline{A}\delta_t}, \tag{6}$$

where

$$\overline{A} = A - B\overline{K}, \quad \text{with } \overline{K} \in \mathbb{R}^{p\times n} : \sigma(\overline{A}) = L. \tag{7}$$

We will prove in Theorem 2 that the feedback gain (6) solves the PSA problem for the system (3), i.e. $L \subset \mathcal{M}\big(A, -BK(\delta_t)\big)$ for any delay function δ_t. Moreover, Theorem 1 states that for a given desired decay rate $\alpha > 0$, such that $\mu(L) < -\alpha$, the feedback gain (6) achieves α-exp stability for the system (3) for any $\delta_t \in [0, \bar{\delta}]$ as long as $\bar{\delta}$ is such that:

$$\int_0^{\bar{\delta}} \|\,\overline{K}e^{\overline{A}t}B\|e^{\alpha t}dt < 1. \tag{8}$$