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Application of differential transform method to non-linear oscillatory systems

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Abstract

In this paper, the differential transform method is proposed for solving non-linear oscillatory systems. These solutions do not exhibit periodicity, which is the characteristic of oscillatory systems. A modification of the differential transform method, based on the use of Padé approximants, is proposed. We use alternative technique by which the solution obtained by the differential transform method is made periodic. The method is described and illustrated with examples. The results reveal that the method is very effective and convenient.

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Keywords: Differential transform; Padé approximation; Laplace transform; Oscillatory equations

1. Introduction

The differential transform is an analytic method for solving differential equations. The concept of the differential transform was first introduced by Zhou in 1986 [1]. Its main application therein is to solve both linear and non-linear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method. The Taylor series method is computationally expensive for large orders. The differential transform method is an alterative procedure for obtaining analytic Taylor series solution of the differential equations. By using DTM, we get a series solution, in practice a truncated series solution. The series often coincides with the Taylor expansion of the true solution at point $x = 0$, in the initial value case. Although the series can be rapidly convergent in a very small region, it has very slow convergence rate in the wider region we examine, and the truncated series solution is an inaccurate solution in that region, which will greatly restrict the application area of the method. Following [2,3] we propose the so-called aftertreatment technique (AT) to modify the differential transform series solution for general ordinary differential equations with initial conditions by using the Padé approxi-

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mant. We will use Laplace transform and Padé approximant to deal with the truncated series. Padé approximant [4] approximates a function by the ratio of two polynomials. The coefficients of the powers occurring in the polynomials are determined by the coefficients in the Taylor series expansion of the function. Generally, the Padé approximant can enlarge the convergence domain of the truncated Taylor series and can improve greatly the convergence rate of the truncated Maclaurin series.

2. Differential transform method

Differential transformation of function $y(x)$ is defined as follows [1]

$$Y(k) = \frac{1}{k!}\left[\frac{\mathrm{d}^k y(x)}{\mathrm{d}x^k}\right]_{x=0}. \tag{1}$$

In (1), $y(x)$ is the original function and $Y(k)$ is the transformed function. Differential inverse transform of $Y(k)$ is defined as follows

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k). \tag{2}$$

In fact, from (1) and (2), we obtain

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{\mathrm{d}^k y(x)}{\mathrm{d}x^k}\right]_{x=0}. \tag{3}$$

Eq. (3) implies that the concept of differential transformation is derived from the Taylor series expansion. From the definitions (1) and (2), it is easy to obtain the following mathematical operations:

1. If $f(x) = g(x) \pm h(x)$, then $F(k) = G(k) + H(k)$.
2. If $f(x) = cg(x)$, then $F(k) = cG(k)$, c is a constant.
3. If $f(x) = \frac{\mathrm{d}^n g(x)}{\mathrm{d}x^n}$, then $F(k) = \frac{(k+n)!}{k!}G(k+n)$.
4. If $f(x) = g(x)h(x)$, then $F(k) = \sum_{l=0}^{k} G(l)H(k-l)$.
5. If $f(x) = x^n$, then $F(k) = \delta(k-n)$, δ is the Kronecker delta.
6. If $f(x) = \int_0^x g(t)\mathrm{d}t$, then $F(k) = \frac{G(k-1)}{k}$, where $k \geqslant 1$.
7. If $f(x) = g(t)\int_0^x h(t)\mathrm{d}t$, then $F(k) = \sum_{l=1}^{k} G(k-l)\frac{H(l-1)}{l}$, where $k \geqslant 1$.
8. If $f(x) = u(x)v(x)w(x)$, then $F(k) = \sum_{s=0}^{k}\sum_{m=0}^{k-s} U(s)V(m)W(k-s-m)$.

Example 1 (*The Duffing equation*).
Consider the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + y + \epsilon y^3 = 0, \tag{4}$$

the initial conditions are chosen to be $y(0) = a$ and $y'(0) = 0$. Taking differential transform of (4), we obtain

$$(k+2)(k+1)Y(k+2) + Y(k) + \epsilon\sum_{s=0}^{k}\sum_{m=0}^{k-s} Y(s)Y(m)Y(k-s-m) = 0, \tag{5}$$

where $Y(k)$ is the differential transform of $y(t)$ and the transform of the initial conditions are $Y(0) = a$ and $Y(1) = 0$. By using Eq. (5) and the transformed initial condition, the following solution is evaluated using MATHEMATICA up to x^6

$$y(t) = a - a(1+\epsilon a^2)\frac{t^2}{2} + a(1+\epsilon a^2)(1+3\epsilon a^2)\frac{t^4}{24} - a(1+\epsilon a^2)(1+24\epsilon a^2+27\epsilon^2 a^4)\frac{t^6}{720}. \tag{6}$$