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Polytopic uncertainty for linear systems: New and old complexity results

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ABSTRACT

We survey the problem of deciding the stability or stabilizability of uncertain linear systems whose region of uncertainty is a polytope. This natural setting has applications in many fields of applied science, from control theory to systems engineering to biology. We focus on the algorithmic decidability of this property when one is given a particular polytope. This setting gives rise to several different algorithmic questions, depending on the nature of time (discrete/continuous), the property asked (stability/stabilizability), or the type of uncertainty (fixed/switching). Several of these questions have been answered in the literature in the last thirty years. We point out the ones that have remained open, and we answer all of them, except one which we raise as an open question. In all the cases, the results are negative in the sense that the questions are NP-hard. As a byproduct, we obtain complexity results for several other matrix problems in systems and control.

1. Introduction

Robust control is an important topic that has motivated several important research lines in systems and control since the eighties. It has spanned a wide range of applications and has benefited from many different techniques in applied mathematics, such as algorithmic complexity, convex optimization, game theory, μ-analysis, and others (see, e.g., [1,2]).

One of the simplest settings in robust control is a discrete-time (resp. continuous-time) linear system describing the evolution of a state space vector $x \in \mathbb{R}^n$ as follows:

$$x(t+1) = A(t)x(t), \quad (1)$$

$$\dot{x}(t) = A(t)x(t), \quad (2)$$

where the matrix $A(t)$ is restricted to belong to a given polytope[1] $P \subset \mathbb{R}^{n\times n}$. Depending on the context, several different questions may be relevant. In some situations, the matrix is fixed for the whole trajectory, but its actual value is not determined, except for the fact that it belongs to the polytope P. We refer to this case as the *fixed uncertainty* case. In other situations the matrix $A(t)$ is allowed to change from time to time; the trajectory of the system is determined by a *switching signal* σ:

$$\sigma : \mathbb{N} \to P : \; t \to A(t), \quad (3)$$

or, in the continuous-time case,

$$\sigma : \mathbb{R}_+ \to P : \; t \to A(t). \quad (4)$$

We refer to this case as the *switching uncertainty* case. Typical questions that one would like to answer are as follows:

Problem 1 (*Stabilizability*)**.** Given a set of matrices $\{A_1, \ldots, A_m\}$ describing a polytope

$$P = \mathrm{Conv}\{A_1, \ldots, A_m\},$$

does there exist a matrix $A \in P$ (resp. a switching signal σ) such that the trajectory converges to zero for any initial condition?

Problem 2 (*Stability*)**.** Given a set of matrices $\{A_1, \ldots, A_m\}$ describing a polytope

$$P = \mathrm{Conv}\{A_1, \ldots, A_m\},$$

does the trajectory converge to zero for every possible matrix $A \in P$ (resp. every switching signal σ) and every initial condition?

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1 Note that polytopes are one of the simplest representations of compact sets, and thus the negative results presented in this paper are generalizable to more complex compact sets.

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Thus, the above setting raises eight different algorithmic questions, depending on the discrete/continuous nature of time, the stability/stabilizability question, and the fixed/switching nature of uncertainty. These questions are typical of *Robust Control*, where one approximatively knows the dynamical system and wants to ensure that it is stable, up to a certain perturbation of the model. Many situations in practical applications boil down to one of these cases (see, e.g., [3]).

A case of particular interest is the analysis of *biological systems* [4,5,2]. There, robustness can be needed because the system is a linearization of a real nonlinear system, and one must then take into account the discrepancy between the model and the real-life situation, or it can be required because the system is not perfectly known. Robustness also arises naturally in neural networks, for uncertainty reasons, or because of the switching nature of their dynamics. The stabilizability problem also turns out to be relevant in situations where one can control a system by activating a certain mode, among a few ones that are available, in order to stabilize the system. For instance, in *virology*, it has been reported that the drug treatment for some viral disease like HIV could be improved by switching among several medications from time to time [6]. Recently, researchers in the control community have modeled this situation as a switching system, where vector $x(t)$ represents the different concentrations of viral populations in the blood, and the switching signal corresponds to the choice of medication at every particular time [7,8]. Typically, in this application, one's goal is to design this switching signal so as to best control the population of viruses in the patient's body. This situation falls into the scope of the present paper, as it can be modeled as a *continuous-time stabilizability problem with switching uncertainty*. Other applications can be found in fields as diverse as *wireless control networks* [9,10], *fluid dynamics* [11], or even *medicine* [12].

In the present paper we focus on the algorithmic problem of answering the above questions. It turns out that all the cases that are known are NP-hard [13–15]. However, it is important to mention that practical algorithms have been developed for some of these problems, which try to circumvent these negative results and which provide practitioners with workable methods that often reach satisfactory answers in practice. For instance, sufficient conditions have been proposed that are efficiently checkable (e.g. in polynomial time) thanks to modern tools like convex optimization methods for proving stability, or instability, of diverse types of uncertain sets of dynamical systems. We refer to [16–24] and references therein for various examples of such methods. While such methods are of course important because they allow us in favorable situations to obtain a solution to the problem, the negative results we present in this paper are also very valuable, as they allow us to understand theoretical barriers that no algorithm could overcome (unless, of course, $P = NP$). Such negative results can also assist in the development of new methods, as they help to understand what type of performance one can hope for in any candidate new method.

In Fig. 1[2] we summarize all known results in the literature (including the ones we derive in this paper). One can see that all the cases are now known to be NP-hard, except for the stabilizability of continuous-time systems with switching uncertainty. In Section 4, we state a conjecture which (if answered positively) would solve the problem.

[2] We note that in [13], a slightly different problem is analyzed, in which the matrix $A(t)$ is restricted to be a vertex of the polytope P. The problem with arbitrary switching in the entire polytope certainly makes sense for this particular question too, but we are not aware of any analysis of this latter problem in the literature.

	Stabilizability		Stability	
	Switched	Fixed	Switched	Fixed
Continuous-time	?	This paper	[39]	[39]
Discrete-time	[38]^2	This paper	[40]	This paper

Fig. 1. Summary of the known results (superscript 2 mentioned in the artwork is explained in Footnote 2).

2. Discrete-time systems with fixed uncertainty

In this section we analyze the stabilizability and stability problems for discrete-time systems with fixed uncertainty. These translate to the problems of asking, for a given matrix polytope, whether there exists a matrix in the polytope with spectral radius smaller than one, or larger than one, respectively. Both problems turn out to be NP-hard.

2.1. Stabilizability

Problem 3 (POLYTOPE-MIN-RADIUS)**.** Given a set of real matrices, is there a convex combination of those whose spectral radius is smaller than one?

Theorem 1. *The* POLYTOPE-MIN-RADIUS *problem is NP-hard.*

Proof. We establish a polynomial-time reduction from the INDEPENDENT-SET problem. This problem asks, for a given undirected graph $\mathcal{G} = (V, E)$ and a positive integer $j \leq |V|$, whether $\mathcal{G}$ contains an independent set V' (a set of pairwise non-adjacent vertices) with size $|V'| \geq j$. This problem is NP-complete [25]. We assume $j \geq 2$ (otherwise the problem is trivial).

An instance of POLYTOPE-MIN-RADIUS takes as input k real $n \times n$ matrices A_i, with $i = 1, \dots, k$, and asks whether there exists a nonnegative vector $\pi = (\pi_1, \dots, \pi_k)^\top$ with $\sum_{i=1}^k \pi_i = 1$, such that the matrix $B_\pi = \sum_{i=1}^k \pi_i A_i$ has spectral radius less than one. We will prove NP-hardness of the problem for the special case in which $k = n$, that is, when the number of input matrices equals their dimension. For any input instance of INDEPENDENT-SET, our reduction will construct (in a number of steps at most polynomial in the size of the problem input) an instance of POLYTOPE-MIN-RADIUS, such that a polynomial-time algorithm for deciding the latter would imply a polynomial-time for deciding the former.

Let $(\mathcal{G}, j)$ be an instance of INDEPENDENT-SET, and let C be the $n \times n$ adjacency matrix of the graph $\mathcal{G}$. The matrix C is a symmetric zero–one matrix with zeros in the main diagonal. Let c_i denote the i'th column of C, and let e_i denote the length-n vector with 1 in the i'th entry and all other entries zero. The reduction constructs n nonnegative block matrices A_i, for $i = 1, \dots, n$, of size $(n + 1) \times (n + 1)$:

$$A_i = \begin{bmatrix} \emptyset & e_i + c_i \\ e_i^\top & r \end{bmatrix}, \tag{5}$$

where $r = 1 - \frac{1}{j-1} \in [0, 1)$ and $\emptyset$ denotes the $n \times n$ zero matrix. The matrix $B_\pi = \sum_{i=1}^k \pi_i A_i$ then reads

$$B_\pi = \begin{bmatrix} \emptyset & (I + C)\pi \\ \pi^\top & r \end{bmatrix}. \tag{6}$$

The special block structure of B_π allows us to analytically compute its eigenvalues by manipulating the system $B_\pi v = \lambda v$, and it is