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Discrepancy principle for DSM II

A.G. Ramm

Mathematics Department, Kansas State University, Manhattan, KS 66506-2602, USA

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Abstract

Let $Ay = f$, A is a linear operator in a Hilbert space H, $y \perp N(A) := \{u : Au = 0\}$, $R(A) := \{h : h = Au, u \in D(A)\}$ is not closed, $\|f_\delta - f\| \leqslant \delta$. Given f_δ, one wants to construct u_δ such that $\lim_{\delta\to 0}\|u_\delta - y\| = 0$. Two versions of discrepancy principles for the DSM (dynamical systems method) for finding the stopping time and calculating the stable solution u_δ to the original equation $Ay = f$ are formulated and mathematically justified.

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1. Introduction

Let A be a linear bounded operator in a Hilbert space H (or in a Banach space X), and equation

$$Au = f \tag{1}$$

be solvable, possibly non-uniquely. Let $N(A) = N$ and $R(A)$ denote the null-space and the range of A, respectively. Denote by y the (unique) minimal-norm solution to (1), $y \perp N$. Given f_δ, $\|f_\delta - f\| \leqslant \delta$, one wants to find a stable approximation u_δ to y:

$$\lim_{\delta\to 0}\|u_\delta - y\| = 0. \tag{2}$$

There are many ways to do this: variational regularization, quasi-solutions, iterative regularization (see e.g., [1,3]).

In [4] a version of the discrepancy principle for DSM was proved. This version consisted in solving the equation for t:

E-mail address: ramm@math.ksu.edu

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$$||T_{a(t)}^{-1}A^*f_\delta - f_\delta|| = c\delta,$$

where $c = \text{const} \in (1,2)$, and $a(t) > 0$ was monotonically decaying and satisfied the assumption:

$$\lim_{t\to\infty} \sup_{\frac{t}{2}\leqslant s\leqslant t} |\dot{a}(s)|a^{-2}(t) = 0.$$

Here we relax the assumptions on $a(t)$ and make the principle easier to apply numerically.

We study a new version of the dynamical systems method (DSM) for finding u_δ:

$$\dot{u}_\delta(t) = -u_\delta(t) + T_{a(t)}^{-1}A^*f_\delta, \quad u_\delta(0) = u_0, \tag{3}$$

where $T := A^*A$ is self-adjoint, $T_a := T + aI$, I is the identity operator,

$$0 < a(t); \quad a(t) \searrow 0 \quad \text{as } t \to \infty; \quad \lim_{t\to\infty} \frac{\dot{a}}{a} = 0, \quad \dot{a} := \frac{\mathrm{d}a}{\mathrm{d}t}. \tag{4}$$

The element u_δ in (2) is $u_\delta(t_\delta)$, where $u_\delta(t)$ is the solution to (3), and t_δ, the stopping time, is found from the following equation for the unknown t:

$$\int_0^t \mathrm{e}^{-(t-s)}a(s)\|Q_{a(s)}^{-1}f_\delta\|\mathrm{d}s = c\delta, \quad c \in (1,2), \tag{5}$$

where $Q := AA^*$ is self-adjoint, $Q_a = Q + aI$, c is a constant, and $\|f_\delta\| > c\delta$. This equation we call a discrepancy principle. About other versions of discrepancy principles see [2–8].

The main result of this paper is the following theorem.

Theorem 1. *Assume that (4) holds, $\|f_\delta\| > c\delta$, and*

$$\lim_{t\to\infty} \mathrm{e}^t a(t)\|Q_{a(t)}^{-1}f_\delta\| = \infty. \tag{6}$$

Then Eq. (5) has a unique solution t_δ,

$$\lim_{\delta\to 0} t_\delta = \infty \tag{7}$$

and (2) holds with $u_\delta := u_\delta(t_\delta)$.

Remark 1. Assumption (6) is always satisfied if $f_\delta \notin R(A)$. Indeed,

$$\lim_{a\to 0} \|aQ_a^{-1}f_\delta\|^2 = \lim_{a\to 0} \int_0^{\|Q\|} \frac{a^2\mathrm{d}(E_\lambda f_\delta, f_\delta)}{(a+\lambda)^2} = \|Pf_\delta\|^2 > 0,$$

where E_λ is the resolution of the identity, corresponding to the self-adjoint operator Q, P is the orthoprojector onto the null-space $N(Q)$ of Q, $N(Q) = N(AA^*) = N(A^*) := N^*$, and $\|P_{N^*}f_\delta\| > 0$ if $f_\delta \notin R(A)$, because $\overline{R(A)} = (N^*)^\perp$.

In Section 2, we prove Theorems 1 and 2, which says that (2) holds without assumption (6) but with an extra assumption $\lim_{t\to\infty} \frac{\dot{a}(t)}{a^2(t)} = 0$.

2. Proofs

Let

$$h(t) := a(t)\|A_{a(t)}^{-1}f_\delta\| := a(t)g(t).$$

Lemma 1. *Assume (6). Then*

$$\lim_{t\to\infty} \frac{\mathrm{e}^t h(t)}{\int_0^t \mathrm{e}^s h(s)\mathrm{d}s} = 1 \tag{8}$$