Chemometrics and Intelligent Laboratory Systems 146 (2015) 329–336

Contents lists available at ScienceDirect

Chemometrics and Intelligent Laboratory Systems

journal homepage: www.elsevier.com/locate/chemolab

Weighted power–weakness ratio for multi-criteria decision making

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ARTICLE INFO

Article history:
Received 28 April 2015
Received in revised form 30 May 2015
Accepted 11 June 2015
Available online 19 June 2015

Keywords:
Power–weakness ratio
PWR
Weighted PWR
Multi-criteria decision making
MCDM

ABSTRACT

This work presents the weighted power–weakness ratio (wPWR), a multivariate index for multi-criteria decision-making (MCDM) and ranking comparison. The index derives from the power–weakness ratio (PWR) originally proposed to select the strongest winner of tournaments, which has been re-adapted in this study to solve MCDM problems. Key features of wPWR are: (1) its multivariate character, (2) the ability to account simultaneously for the strengths and weaknesses of each element, (3) the possibility to weight criteria according to previous knowledge about the problem. In order to analyze wPWR, we selected three datasets available in scientific literature. The obtained wPWR scores and rankings were compared with four well-established techniques: Simple Average Ranking, Dominance, Reciprocal Rank Fusion and Kendall–Wei approach. Where rankings obtained by other techniques were available from literature, they were also included in the analysis. Results highlighted a correct correlation between wPWR and the other ranking measures, but also interesting differences that support its introduction in the field of MCDM and chemometrics.

1. Introduction

Multi-criteria decision making (MCDM) refers to making a systematic and rational decision of the best alternative among several candidates when multiple (and often conflicting) criteria are present [1]. MCDM methods help to solve complex decision-making challenges, which can comprise up to thousands of candidates, represented by as many criteria. Throughout the years, many MCDM techniques have been proposed [2], which had successful applications in numerous fields, such as environmental management [3], risk assessment [4], energy planning [5], finance [6], health care [7], research evaluation [8] and find useful applications in everyday life [9], as well.

A crucial aspect of multi-criteria approaches, which can strongly affect the outcome of the decision process, is the relative importance given to each criterion in determining the final decision. This is numerically expressed by the so-called weights, which are determinable by different rules [10]. When the possibility to weight the criteria on the basis of previous knowledge is given, a more flexible and rational application of MCDM techniques to complex problems is allowed for.

In the present work, we introduce a novel approach for MCDM, using a weighted version of the power–weakness ratio (PWR), originally proposed by Ramanujacharyulu in 1964 [11]. PWR was initially thought to select the winner in tournaments or the most influential person within a group. It is based on the idea of taking into account simultaneously the candidate's “power” (e.g. strength of players he won against) and his “weakness” (e.g. strength of players he lost against) to obtain the final ranking. PWR found later applications to round robin tournaments [12] and contestants evaluation [13]. Recently, PWR was used to analyze the results of the English Premier League football tournament [14] and was proposed as journal indicator [15]. Despite the simplicity of the method, PWR was never used for chemometric applications.

Our work stems from the idea of adapting PWR from ranking problems, where the criteria all have the same meaning (e.g. matches of a tournament, people of a group), to MCDM problems where the criteria can have different meanings and relevance. In this way, we obtained what we called weighted PWR (wPWR), which is a generalized version of the original PWR and is suitable to solve complex MCDM problems as well as to compare rankings. Moreover, wPWR is characterized by a multivariate and holistic approach, which, differently from other existing MCDM methods, is able to account simultaneously for all the interactions between the alternatives, offering insights about data structure.

In this paper, after introducing wPWR, we compared it with some well-known MCDM approaches (e.g. dominance, average ranking) in order to investigate its potential in chemometrics and decision-making issues.

2. Theory

2.1. Power–weakness ratio (PWR)

The power–weakness ratio (PWR) was proposed in 1964 by Ramanujacharyulu [11] as a method to find the winner of a tournament or the most influential person within a group. PWR aims to locate the most talented individual, defined as the one that won over the largest

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http://dx.doi.org/10.1016/j.chemolab.2015.06.005
0169-7439/

number of opponents (maximum “power”) and simultaneously was defeated by few opponents (minimum “weakness”). PWR was hence thought to capture the balance between the power and the weakness of each individual.

Let us suppose that we are considering n individuals participating to a tournament. The tournament table, **T**, is the $n \times n$ matrix of pairwise comparisons, where each cell t_{ij} represents how many times the individual i won over the jth individual. This matrix encodes the power of any player over the remaining $n - 1$ ones.

On the contrary, the transpose of the tournament table, $\mathbf{T}^\mathbf{T}$, contains information about the weakness of each individual. Each element of the matrix (t^*_{ij}) is a count of how many times the ith player was defeated by the jth player. The properties of both these tournament matrices have been studied extensively elsewhere (e.g. [16,17]).

Kendall and Wei [18,19] firstly proposed a method to obtain a ranking from **T**. They suggested to raise iteratively each element of **T** to the power of b ($b \in \mathbb{N}$) and then rank the players according to the obtained row sums (Kendall–Wei scores). This allows accounting for higher order degrees of interaction between players. In other words, for each chosen b, the bth order of power of each player is taken into account. Naturally, the player that possesses the largest power of order b is the talented person at the bth stage. Thus, if we consider the limit $b \to \infty$, we will be able to select the most talented player above all.

Tournament matrices are asymmetrical, nonnegative and irreducible. According to the Perron–Frobenius (PF) [20] theorem, we know that:

$$\lim_{b\to\infty}\left(\frac{\mathbf{T}}{\lambda_{PF}}\right)^b \cdot \mathbf{U} = \mathbf{e}_{PF} \qquad b \in \mathbb{N} \tag{1}$$

where λ_{PF} is the largest eigenvalue of **T**, $\mathbf{e}_{PF}$ is the corresponding eigenvector and **U** is a unity matrix. This theorem ensures the existence of a positive λ_{PF} and of a unique end entrywise positive $\mathbf{e}_{PF}$.

Eq. (1) expresses the convergence of the iterative power of **T** to $\mathbf{e}_{PF}$ for $b \to \infty$. The elements of **T** can be thus ranked according to their values of PF eigenvector, calculated as follows:

$$\mathbf{T}\mathbf{e}_{PF} = \lambda_{PF}\mathbf{e}_{PF}. \tag{2}$$

By ranking the players according to their $\mathbf{e}_{PF}$ entries, those that win more are awarded. In this way, the Kendall–Wei method (KW) awards players that win at most.

Correspondingly, also a ranking on $\mathbf{T}^\mathbf{T}$ can be performed by eigenvalue-eigenvector decomposition, as follows:

$$\mathbf{T}^\mathrm{T}_{PF}\mathbf{e}^*_{PF} = \lambda^*_{PF}\mathbf{e}^*_{PF} \tag{3}$$

where $\mathbf{e}^*_{PF}$ is the PF-eigenvector calculated on $\mathbf{T}^\mathbf{T}$ and λ^*_{PF} the corresponding eigenvalue. In this case, individuals that were defeated by few players will have small loadings values. The best player, when $\mathbf{T}^\mathbf{T}$ is considered, is that suffering the fewest losses. It is important to note that ranking the element according to λ^*_{PF} not necessarily gives the same ranking as that obtained by using λ_{PF}.

The power–weakness ratio (PWR) takes into account both the information encoded within **T** and $\mathbf{T}^\mathbf{T}$. For each ith player it is defined as follows:

$$\mathrm{PWR}_i = \frac{e_i}{e^*_i} \tag{4}$$

where e_i is the entry of the ith element on the PF-eigenvector obtained from **T** ($\mathbf{e}_{PF}$), while e^*_i is that obtained from the PF-eigenvector of $\mathbf{T}^\mathbf{T}$ ($\mathbf{e}^*_{PF}$). PWR will be higher for those players that won more against strong players and lost less against weak players; this is allowed by the multivariate character of this index.

Table 1
Tournament table $\mathbf{T_1}$. For each t_{ij}: 1 if the P_i player won over P_j, 0 if P_j won, 0.5 if they drew the match.

$\mathbf{T_1}$	P1	P2	P3	P4	P5
P1	0	1	1	0.5	0
P2	0	0	1	0.5	0.5
P3	0	0	0	1	1
P4	0.5	0.5	0	0	0.5
P5	1	0.5	0	0.5	0

2.1.1. Example 1

Table 1 reports a tournament table ($\mathbf{T_1}$), collecting the results of five players after the first turn (i.e. four matches for each player). Scores of each ith row and the jth column are defined as follows: 1 if player i defeated player j; 0 if player i was defeated by j; 0.5 if they drew the match.

From the tournament table, the Perron–Frobenius eigenvector and the corresponding PWR values for each player were obtained (Table 2). The resulting ranking is P1 > P3 > P5 > P2 > P4.

In case of a second turn having the same results of the first (i.e. $[\mathbf{T_2}]_{ij} = 2 \cdot [\mathbf{T_1}]_{ij}$), the obtained PWR and PF-eigenvectors would be the same as those obtained by $\mathbf{T_1}$, since the strength/weakness of each player is the same. The only difference would be between the eigenvalues of $\mathbf{T_1}$ and $\mathbf{T_2}$, the latter being twice as the former (1.963 and 3.926, respectively).

2.2. Proportional scoring for PWR

The PWR in its original formulation is sensitive to the number of matches played by each individual. In case of unequal number of matches played, the PWR will be biased toward those individuals that played (and won) the largest number of games.

In this work, we propose an alternative way to calculate PWR, useful when the number of matches played is not equal among the individuals. It consists in adopting a proportional scoring, obtaining a tournament table ($\mathbf{T}^\mathrm{P}$) whose elements are scaled as follows:

$$t^\mathrm{P}_{ij} = \frac{t_{ij}}{t_{ij} + t_{ji}} \tag{5}$$

When the players did not play against each other, $t^\mathrm{P}_{ij} = t^\mathrm{P}_{ji} = 0.5$ is assumed. In case of unequal number of matches, when $\mathbf{T}^\mathrm{P}$ is used instead of **T**, the calculated PWR is less sensitive to the number of games played by each individual (see Example 2). In case of equal number of matches, the PWR calculated on $\mathbf{T}^\mathrm{P}$ is the same as that obtained by **T**.

2.2.1. Example 2

For the second example we used a historic record of a tennis tournament of 1975 between four players [13]. In this case, players did not play the same number of matches against each other (Fig. 1). By calculating PWR on the original data, the original table ($\mathbf{T_1}$) and the proportional table ($\mathbf{T}^\mathrm{P}_1$) lead to different scores (Fig. 1c) but to the same rankings (Fig. 1d). If the number of matches won by one player changes significantly (Fig. 1e), the PWR calculated on $\mathbf{T_2}$ and the corresponding

Table 2
Results of PWR scoring on table $\mathbf{T_1}$. Entries of the Perron–Frobenius eigenvector calculated on tournament table ($\mathbf{e}_{PF}$) and on its transpose ($\mathbf{e}^*_{PF}$) for each player are also reported.

Players	e_{PF}	$e_{PF}{}^*$	PWR
P1	0.529	0.370	1.368
P2	0.430	0.442	0.976
P3	0.426	0.414	1.025
P4	0.364	0.535	0.714
P5	0.471	0.459	1.023