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# Stability properties of a class of positive switched systems with rank one difference

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ABSTRACT

Given a single-input continuous-time positive system, described by a pair $(A, \mathbf{b})$, with $A$ a diagonal matrix, we investigate under what conditions there exists a state-feedback law $u(t) = \mathbf{c}^\top \mathbf{x}(t)$ that makes the resulting controlled system positive and asymptotically stable, by this meaning that $A + \mathbf{b}\mathbf{c}^\top$ is Metzler and Hurwitz. In the second part of this note we assume that the state-space model switches among different state-feedback laws ($\mathbf{c}_i^\top$, $i = 1, 2, \dots, p$) each of them ensuring the positivity, and show that the asymptotic stability of this type of switched system is equivalent to the asymptotic stability of all its subsystems, while its stabilizability is equivalent to the existence of an asymptotically stable subsystem.


## 1. Introduction

Recent years have seen a growing interest in systems that are subject to a positivity constraint on their dynamical variables. There are several motivations for this interest, coming from different domains of science and technology. In fact, the positivity assumption is a natural one when describing physical, biological or economical processes whose variables represent quantities that are intrinsically nonnegative, such as pressures, concentrations, population levels, etc. [1].

By a continuous-time positive switched system (CPSS) we mean a dynamic system consisting of a family of continuous-time positive state-space models and a switching law, specifying when and how the switching takes place. CPSS have been fruitfully used in bioengineering and pharmacokinetics. For instance, the insulin–sugar metabolism is captured by two different compartmental models: one valid in steady-state and the other (of course, more complex) which is suitable to describe the evolution under perturbed conditions, following an oral consumption or an intravenous injection. The paper by Haddad, Chellaboina and Nersesov [2] provides a very interesting analysis of hybrid nonnegative systems and, in particular, of hybrid compartmental systems and their use in modeling physiological systems.

In intracellular systems biology, the continuous time dynamics of signaling pathways are often combined with the essentially logical machinery of gene expression. Together with transport delays in protein synthesis, this may lead to hybrid (in particular, switched) systems with time delays and positivity constraints on the describing variables [3]. Positive switched systems have also been used to design optimal drug treatments to cope with viral mutation [4].

CPSSs have been the object of an intense research activity, mainly focused on stability [5–12] and stabilizability [13–15]. Special attention has been devoted to the class of CPSSs that switch among subsystems whose matrices differ by a rank one matrix [16,10,11,17,18]. The reason for the interest in these systems is twofold. On the one hand, they can be thought of as the possible configurations one obtains from a given single-input system, when applying different state-feedback laws that ensure the positivity of the resulting closed-loop system. For this reason, the subsystem matrices can be denoted by $A + \mathbf{b}\mathbf{c}_i^\top$, $i \in \{1, 2, \dots, p\}$. On the other hand, interesting connections have been highlighted [18] between the quadratic stability of CPSSs, switching between two subsystems of matrices $A$ and $A+\mathbf{b}\mathbf{c}^\top$, and the SISO circle criterion for the transfer function $\mathbf{c}^\top(sI_n - A)^{-1}\mathbf{b}$.

In [11] it has been proved that, when a CPSS switches between $p = 2$ subsystems of dimension $n \leq 3$, the Hurwitz property of its subsystem matrices $A+\mathbf{b}\mathbf{c}_i^\top$, $i \in \{1, 2\}$, ensures the asymptotic stability of the associated CPSS. On the other hand, as one deduces by putting together the results of [8,19], this is also true when the CPSS has dimension $n = 2$ and consists of an arbitrary number of subsystems. At the present stage of research, it is not known whether the Metzler Hurwitz property of the matrices $A+\mathbf{b}\mathbf{c}_i^\top$, $i \in \{1, 2, \dots, p\}$, $p > 2$ of size $n > 2$ ensures that the associated CPSS is asymptotically stable. In this paper we prove that this is true

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under the additional assumption that the matrix $A$ is a diagonal one.

CPSSs described by Metzler matrices $A+\mathbf{b}\mathbf{c}_i^\top$, $i\in\{1,2,\dots,p\}$, with $A$ diagonal, arise when investigating the behavior of non-homogeneous multi-agent systems, each of them described by a scalar system, evolving under the action of a unique input signal, that coordinates their behavior. If we assume that different state-feedback strategies may be employed to control the overall agent behavior, we naturally end up with this class of rank one CPSSs, having a diagonal system matrix. This kind of model arises also when dealing with compartmental models, with independent compartments, that are subject to different supervisory control strategies (e.g., tracers injections whose quantities depend on a weighted sum of the compartment concentrations, as it happens with some drug treatments).

In addition, the stability result derived in this paper is relevant also for non-positive switched systems whose subsystem matrices differ by a rank one matrix. Indeed, if we drop the positivity constraint, it is known that the Hurwitz property of the subsystem matrices alone does not ensure the asymptotic stability of the associated switched system, and additional conditions are required [20]. However, from the aforementioned result it follows that when $A$ is diagonalizable, and the matrices $A+\mathbf{b}\mathbf{c}_i^\top$ leave invariant the polyhedral cone generated by $n$ (linearly independent, but otherwise arbitrarily chosen) eigenvectors of $A$, then the Hurwitz property of the matrices $A+\mathbf{b}\mathbf{c}_i^\top$ ensures the asymptotic stability of the switched system. It is conjectured that the existence of a proper polyhedral cone, left invariant by all the Hurwitz matrices $A+\mathbf{b}\mathbf{c}_i^\top$, may lead to obtain a complete characterization of the asymptotic stability property in the non-positive case.

In the second part of the paper, stabilizability of CPSSs with rank one difference, under the assumption that $A$ is diagonal, is shown to be equivalent to the asymptotic stability of at least one subsystem (i.e., existence of an index $i$ such that $A+\mathbf{b}\mathbf{c}_i^\top$ is Hurwitz). While the sufficiency of this condition is obvious, its necessity is not, and essentially reveals that no smart switching strategy may overcome the drawback related to the fact that all subsystems are not asymptotically stable. Note that stabilizability of CPSSs that switch among subsystems whose matrices differ by a rank one matrix has not been addressed before in the literature, except in [19], where the main focus, however, is on convex combinations of the subsystem matrices.

In detail, the paper is organized as follows: in Section 2, we present some preliminary results and consider a continuous-time single-input state-space model with diagonal system matrix $A$. Conditions on the vectors $\mathbf{b}$ and $\mathbf{c}$ that ensure the positivity and the asymptotic stability of the resulting system $\dot{\mathbf{x}}(t)=(A+\mathbf{b}\mathbf{c}^\top)\mathbf{x}(t)$ are provided. Section 3 solves the stability problem of the class of rank one CPSSs, while stabilizability is the object of Section 4. A preliminary version of part of the results appearing in this paper was recently presented at the ECC 2013 Conference [21].

*Notation.* $\mathbb{R}_+$ is the semiring of nonnegative real numbers and, for any pair of positive integers $k,n$, with $k\le n$, $[k,n]$ is the set of integers $\{k,k+1,\dots,n\}$. The $i$th entry of a vector $\mathbf{v}$ is denoted by $[\mathbf{v}]_i$. We denote by $\mathbf{1}_n$ the $n$-dimensional vector with all unitary entries, and by $\mathbf{e}_i$ the $i$th canonical vector in $\mathbb{R}^n$ ($n$ being clear from the context), with all zero entries except for the $i$th which is unitary. A matrix (in particular, a vector) $A$ with entries in $\mathbb{R}_+$ is called *nonnegative*, and if so we adopt the notation $A\ge 0$. If, in addition, $A$ has at least one positive entry, the matrix is *positive* ($A>0$), while if all its entries are positive, it is *strictly positive* ($A\gg 0$). A *Metzler matrix* is a real square matrix, whose off-diagonal entries are nonnegative. A square matrix $A$ is *Hurwitz* if all its eigenvalues have negative real part.

## 2. Diagonal systems and positivity preserving stabilizing feedback laws

Consider a single-input state-space model

$$\dot{\mathbf{x}}(t)=A\mathbf{x}(t)+\mathbf{b}u(t),\quad t\in\mathbb{R}_+, \tag{1}$$

where $\mathbf{x}(t)$ and $u(t)$ are the $n$-dimensional state variable and the scalar input, respectively, at time $t$. We assume that $A$ is diagonal, namely $A=\mathrm{diag}\{\lambda_1,\lambda_2,\dots,\lambda_n\}$, $\lambda_i\in\mathbb{R}$.

We consider a state feedback law $u(t)=\mathbf{c}^\top\mathbf{x}(t)$ that makes the resulting autonomous system positive, by this meaning that the matrix $A+\mathbf{b}\mathbf{c}^\top$ is Metzler. It is worth noticing that $A+\mathbf{b}\mathbf{c}^\top$ is Metzler if and only if $\mathbf{b}\mathbf{c}^\top$ is Metzler, and this introduces strong constraints on the sign of the nonzero entries of the vectors $\mathbf{b}$ and $\mathbf{c}$. In particular, if all the entries of the vectors $\mathbf{b},\mathbf{c}\in\mathbb{R}^n$ are nonzero, the product $\mathbf{b}\mathbf{c}^\top$ is Metzler if and only if one of the following applies:

- if $n=1$, $\mathbf{b}$ and $\mathbf{c}$ can be arbitrary;
- if $n=2$, either all entries of $\mathbf{b}$ and $\mathbf{c}$ have the same sign (in which case $\mathbf{b}\mathbf{c}^\top\gg 0$), or both $\mathbf{b}$ and $\mathbf{c}$ have two entries of opposite sign and $\mathbf{b}\mathbf{c}^\top$ has positive off-diagonal entries and negative diagonal entries;
- if $n>2$, then all entries of $\mathbf{b}$ and $\mathbf{c}$ have the same sign (and hence, again $\mathbf{b}\mathbf{c}^\top\gg 0$).

We first explore the eigenvalue allocation problem, namely we investigate where the eigenvalues of the matrix $A+\mathbf{b}\mathbf{c}^\top$ can be located under the assumption that $A$ is diagonal and $A+\mathbf{b}\mathbf{c}^\top$ is Metzler. To this end, it entails no loss of generality reordering the state components in such a way that $A$, $\mathbf{b}$ and $\mathbf{c}$ are block-partitioned (with corresponding blocks having the same size) as follows:

$$A=\begin{bmatrix}A_1&&&\\&A_2&&\\&&A_3&\\&&&A_4\end{bmatrix},\qquad \mathbf{b}=\begin{bmatrix}\mathbf{b}_1\\\mathbf{b}_2\\0\\0\end{bmatrix},\qquad \mathbf{c}=\begin{bmatrix}\mathbf{c}_1\\0\\\mathbf{c}_3\\0\end{bmatrix}, \tag{2}$$

where all the entries of the blocks $\mathbf{b}_1,\mathbf{b}_2,\mathbf{c}_1,\mathbf{c}_3$ are nonzero, and each block $A_i$, $i\in[1,4]$, is diagonal of size $n_i$. This is a simple consequence of fact that the set $[1,n]$ can be partitioned into the four (possibly empty) disjoint sets:

$$I_1:=\{j\in[1,n]:[\mathbf{b}]_j\ne 0\text{ and }[\mathbf{c}]_j\ne 0\},$$
$$I_2:=\{j\in[1,n]:[\mathbf{b}]_j\ne 0\text{ and }[\mathbf{c}]_j=0\},$$
$$I_3:=\{j\in[1,n]:[\mathbf{b}]_j=0\text{ and }[\mathbf{c}]_j\ne 0\},$$
$$I_4:=\{j\in[1,n]:[\mathbf{b}]_j=0\text{ and }[\mathbf{c}]_j=0\}.$$

Moreover, we assume w.l.o.g. that $A_1=\mathrm{blockdiag}\{\tilde\lambda_1 I_{k_1},\tilde\lambda_2 I_{k_2},\dots,\tilde\lambda_r I_{k_r}\}$, with $\tilde\lambda_1>\tilde\lambda_2>\dots>\tilde\lambda_r$.

**Proposition 1.** *Given a diagonal matrix $A\in\mathbb{R}^{n\times n}$, $n>1$, and vectors $\mathbf{b},\mathbf{c}\in\mathbb{R}^n$, described as in (2), assume that the matrix $A+\mathbf{b}\mathbf{c}^\top$ is Metzler. Then*

(i) $\sigma(A+\mathbf{b}\mathbf{c}^\top)=\sigma(A_1+\mathbf{b}_1\mathbf{c}_1^\top)\cup\sigma(A_2)\cup\sigma(A_3)\cup\sigma(A_4)$.

*Moreover, the spectrum $(\mu_1,\mu_2,\dots,\mu_{n_1})$ of $A_1+\mathbf{b}_1\mathbf{c}_1^\top$ satisfies the following conditions:*

(ii) *if $n_1=1$, then $\mu_1=\tilde\lambda_1+\mathbf{b}_1\mathbf{c}_1$;*

(iii) *if $n_1>2$, then $\tilde\lambda_1,\tilde\lambda_2,\dots,\tilde\lambda_r$ are eigenvalues of $A_1+\mathbf{b}_1\mathbf{c}_1^\top$ of multiplicities $k_1-1,k_2-1,\dots,k_r-1$, while the remaining $r$ eigenvalues of $A_1+\mathbf{b}_1\mathbf{c}_1^\top$, say $\mu_1>\mu_2>\dots>\mu_r$, satisfy*

$$\tilde\lambda_r<\mu_r<\tilde\lambda_{r-1}<\mu_{r-1}<\dots<\mu_2<\tilde\lambda_1<\mu_1; \tag{3}$$