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Stochastic maximum principle for optimal control with multiple priors[☆]

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ABSTRACT

The necessary condition is derived for optimal control with multiple priors which are mutually singular. The tool we use is the theory of *G*-expectation.

1. Introduction

The necessary condition for optimal control, also called stochastic maximum principle, is one of the important topics in control theory. A number of studies have been devoted to this topic. Peng [1] proved a general maximum principle for forward stochastic control system using second order duality technique to overcome the difficulty that the control variable entering the diffusion coefficient. Peng [2] firstly studied optimal control for a kind of forward–backward stochastic control system which appears in economics and mathematical finance. Then many works focus on the stochastic maximum principle, see among many others, Li and Tang [3], Wu [4], Cadenillas [5] and references therein.

All the above works consider the stochastic maximum principle under a single linear probability space. The uncertain volatility model [6,7] demonstrates that sometimes we have to work under a set of probability measures, even they are mutually singular to each other. For example, the super hedging problem introduces a sublinear pricing operator which can be seen as the supremum of a set of linear expectations. We now consider a stochastic control system (2.1) and minimize the cost functional (2.2) in which $\mathbb{E}[\cdot|\mathcal{F}_t] : L_G^1(\mathcal{F}_T) \mapsto L_G^1(\mathcal{F}_t)$ is the *G*-expectation [8, Chapter III Section 2], a kind of sublinear expectation possessing the following properties:

(i) Monotonicity: If $X \geq Y$, then $\mathbb{E}[X|\mathcal{F}_t] \geq \mathbb{E}[Y|\mathcal{F}_t]$.
(ii) Constant preserving: $\mathbb{E}[c|\mathcal{F}_t] = c,\ \forall c \in \mathbf{R}$.
(iii) Sub-additivity: $\mathbb{E}[X+Y|\mathcal{F}_t] \leq \mathbb{E}[X|\mathcal{F}_t] + \mathbb{E}[Y|\mathcal{F}_t]$.
(iv) Positive homogeneity: $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X],\ \forall \lambda \geq 0$.
(v) If $\mathbb{E}[X|\mathcal{F}_t] = -\mathbb{E}[-X|\mathcal{F}_t]$, for some t, then $\mathbb{E}[X+Y|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{F}_t] + \mathbb{E}[Y|\mathcal{F}_t]$.
(vi) $\mathbb{E}[X+\eta|\mathcal{F}_t] = \mathbb{E}[X|\mathcal{F}_t] + \eta,\ \eta \in L_G^1(\mathcal{F}_t)$.

Hu and Peng [9] proved the representation of sublinear expectation: $\mathbb{E}[\cdot] = \sup_{P\in\mathcal{P}} E_P[\cdot]$, $\mathcal{P}$ is a set of linear probability which are mutually singular. A property holds "quasi-surely" (q.s.) if it holds outside a polar set A, i.e., $P(A) = 0,\ \forall P \in \mathcal{P}$. Let $(B(t))_{t\geq 0}$ be a *G*-Brownian motion [8, Chapter III Definition 1.2] under $\mathbb{E}$. It is shown that $B(\cdot)$ is a martingale under every $P \in \mathcal{P}$ [10,11] and there exists a unique adapted process (σ_t^P) such that $\underline{\sigma}^2 \leq \left(\sigma_t^P\right)^2 \leq \overline{\sigma}^2$, a.e. t, P-a.s.[1] and

$$B_t = \int_0^t \sigma_s^P dW_s^P, \quad \forall t \geq 0,\ P\text{-a.s.}$$

where $\overline{\sigma}^2 := \mathbb{E}[\langle B\rangle_1]$, $\underline{\sigma}^2 := -\mathbb{E}[-\langle B\rangle_1]$, (W_t^P) is a standard E_P-Brownian motion, $\langle\cdot\rangle_.$ denotes the quadratic variance of a process.

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1 a.e.: almost everywhere; a.s.: almost surely.

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Therefore an interesting phenomenon comes up: the quadratic variance of (B_t) under any $P \in \mathcal{P}$,

$$\langle B\rangle_t = \int_0^t \left|\sigma_s^P\right|^2 ds, \quad \forall t \geq 0,\ P\text{-a.s.}$$

is no longer a deterministic function of time t. It is random. One of the most distinctions between G-stochastic analysis and Itô's calculus comes here: $N_t := \int_0^t \alpha_s d\langle B\rangle_s - \int_0^t 2G(\alpha_s)\, ds$ is a G-martingale while $(-N_t)$ is not. The related function $G : \mathbf{R} \mapsto \mathbf{R}$ is defined by

$$G(a) = \frac{1}{2}\sup_{\underline{\sigma}^2 \leq \gamma \leq \overline{\sigma}^2} \{\gamma a\} = \frac{1}{2}[\overline{\sigma}^2 a^+ - \underline{\sigma}^2 a^-], \quad a \in \mathbf{R}.$$

It is easy to prove that $u(t,x) = \mathbb{E}[\varphi(x + B_t)]$ is the solution of the following G-heat equation

$$\partial_t u(t,x) - G\left(D_x^2 u\right) = 0, \qquad u(0,x) = \varphi(x).$$

Itô's integral with respect to (B_t) [8], Itô's formula [12] and martingale representation [10,13] are all well established in this framework. See [8] for an overview of G-stochastic analysis.

Definition 1.1. (X_t) is a G-martingale if $X_s = \mathbb{E}[X_t|\mathcal{F}_s],\ s \leq t$. If moreover $\mathbb{E}[X_t|\mathcal{F}_s] = -\mathbb{E}[-X_t|\mathcal{F}_s]$, we call (X_t) a symmetric G-martingale.

Itô's integral $\int_0^\cdot Z_s dB_s$ is a symmetric G-martingale. $\langle B\rangle_t - \overline{\sigma}^2 t$ is a G-martingale but not symmetric. For a partition of $[0,T]$: $0 = t_0 < t_1 < \cdots < t_N = T$ and $p \geq 1$, we set

$\mathcal{M}_G^{p,0}(0,T)$: the collection of processes $\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \cdot 1_{[t_j,t_{j+1})}(t)$, where $\xi_j \in L_G^p(\Omega_{t_j}),\ j = 0, 1, \ldots, N$;

$\mathcal{M}_G^p(0,T)$: the completion of $\mathcal{M}_G^{p,0}(0,T)$ under norm $\|\eta\|_{\mathcal{M}} = \left(\mathbb{E}\left[\int_0^T |\eta_t|^p dt\right]\right)^{\frac{1}{p}}$;

$\mathcal{H}_G^p(0,T)$: the completion of $\mathcal{M}_G^{p,0}(0,T)$ under norm $\|\eta\|_{\mathcal{H}} = \left(\mathbb{E}\left(\int_0^T |\eta_t|^2 dt\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$. It is easy to prove that $\mathcal{H}_G^2(0,T) = \mathcal{M}_G^2(0,T)$.

We have the following Martingale Representation theorem from Song [13].

Proposition 1.1. *Let $\underline{\sigma} > 0$. For $\xi \in L_G^\beta(\Omega)$ with some $\beta > 1$, G-martingale $X_t = \mathbb{E}[\xi|\mathcal{F}_t]$, $t \in [0,T]$ has the following unique decomposition:*

$$X_t = X_0 + \int_0^t Z_s dB_s - K_t, \quad q.s. \tag{1.1}$$

where $(Z_t) \in \mathcal{H}_G^\alpha(0,T)$, $K_T \in L_G^\alpha(\Omega)$ for any $1 \leq \alpha < \beta$ and (K_t) is a continuous, increasing process with $K_0 = 0$ and $(-K_t)$ being a G-martingale.

2. Stochastic maximum principle with multiple priors

2.1. Statement of the problem

We consider one dimensional control system. There are no essential difficulties for the multidimensional one.

Let b, σ, l and h be such that

$b(t,x,v), \sigma(t,x,v), l(t,x,v) : [0,T] \times \mathbf{R} \times \mathbf{R} \mapsto \mathbf{R}$,

$h(x) : \mathbf{R} \mapsto \mathbf{R}$.

We assume

- (H1) b, σ, l and h are continuous in $[0,T] \times \mathbf{R} \times \mathbf{R}$ and they are continuously differentiable with respect to (x, v).
- (H2) The derivatives of b, σ are bounded.
- (H3) The derivatives of l are bounded by $C(1 + |x| + |v|)$ and the derivative of h is bounded by $C(1 + |x|)$.

Let U be a nonempty convex subset of $\mathbf{R}$. We define the admissible controls set

$$\mathcal{U} = \left\{v(\cdot) \in \mathcal{M}_G^2(0,T) | v(t) \in U,\ \text{a.e., q.s.}\right\}.$$

Obviously, $\mathcal{U}$ is also a convex set. For any given admissible control $v(\cdot) \in \mathcal{U}$ and initial state $x_0 \in \mathbf{R}$, we consider the following stochastic control system:

$$\begin{aligned} dx(t) &= b(t, x(t), v(t))\, dt + \sigma(t, x(t), v(t))\, dB(t), \\ &\quad t \in [0,T], \\ x(0) &= 0, \end{aligned} \tag{2.1}$$

where $B(\cdot)$ is a G-Brownian motion. It is just a consequence of Peng [8], Ch.V that there is a unique solution $x(\cdot) \in \mathcal{M}_G^2(0,T)$ to Eq. (2.1) on $[0,T]$, $T > 0$ is a fixed time. $x(\cdot)$ is called the state variable or trajectory. The optimal control problem is to minimize the following cost functional over $\mathcal{U}$:

$$J(v(\cdot)) = \mathbb{E}\left[\int_0^T l(t, x(t), v(t))\, dt + h(x(T))\right], \tag{2.2}$$

$$\inf_{v(\cdot)\in\mathcal{U}} J(v(\cdot)), \tag{2.3}$$

where $\mathbb{E}$ is the G-expectation, a sublinear expectation generated by a set of singular probability measures. The classical optimal control deals with one single probability measure. Now we have to work under a set of probability measures which are singular with each other. The main motivation of this kind of optimal control is the uncertain volatility model proposed by [6].

2.2. Variational equation and variational inequality

In order to derive the maximum principle, we use the classical "convex variation method" introduced by Bensoussan [14]. Let $u(\cdot)$ be an optimal control and $x(\cdot)$ be the corresponding trajectory. Let $v(\cdot)$ be such that $u(\cdot) + v(\cdot) \in \mathcal{U}$. Since $\mathcal{U}$ is convex, then for any $\rho \in [0,1]$, $u_\rho(\cdot) = u(\cdot) + \rho v(\cdot)$ is also in $\mathcal{U}$. We denote $x_\rho(\cdot)$ the corresponding trajectory.

Let $\eta(\cdot)$ be the solution of the following variational equation:

$$\begin{aligned} d\eta(t) &= (b_x(t, x(t), u(t))\, \eta(t) + b_v(t, x(t), u(t))\, v(t))\, dt \\ &\quad + (\sigma_x(t, x(t), u(t))\, \eta(t) \\ &\quad + \sigma_v(t, x(t), u(t))\, v(t))\, dB(t), \\ \eta(0) &= 0. \end{aligned} \tag{2.4}$$

By condition (H2) and Peng [8], we can find a unique solution $\eta(\cdot) \in \mathcal{M}_G^2(0,T)$ to Eq. (2.4). Set

$$\tilde{x}_\rho(t) = \rho^{-1}(x_\rho(t) - x(t)) - \eta(t).$$

We have the following convergence result:

Lemma 2.1. *Let* (H1) *and* (H2) *hold. Then*

$$\lim_{\rho\downarrow 0} \sup_{t\in[0,T]} \mathbb{E}\left|\tilde{x}_\rho(t)\right|^2 = 0.$$

Proof. We denote, for simplicity, the subscript t is omitted,

$$A_\rho := \int_0^1 b_x\left(x + \lambda\rho\left(\tilde{x}_\rho + \eta\right), u + \lambda\rho v\right) d\lambda,$$

$$G_{1\rho} := \left[A_\rho - b_x(x,u)\right]\eta + \int_0^1 \left[b_v(x, u + \lambda\rho v) - b_v(x,u)\right] d\lambda,$$

$$B_\rho := \int_0^1 \sigma_x\left(x + \lambda\rho\left(\tilde{x}_\rho + \eta\right), u + \lambda\rho v\right) d\lambda,$$

$$G_{2\rho} := \left[B_\rho - \sigma_x(x,u)\right]\eta + \int_0^1 \left[\sigma_v(x, u + \lambda\rho v) - \sigma_v(x,u)\right] d\lambda.$$