Systems & Control Letters 62 (2013) 311–318

Contents lists available at SciVerse ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Spike controls for elliptic and parabolic PDEs

E. Casas [a,*], E. Zuazua [b,c]

[a] Departamento de Matemática Aplicada y Ciencias de la Computación, E.T.S.I. Industriales y de Telecomunicación, Universidad de Cantabria, 39005 Santander, Spain

[b] Basque Center for Applied Mathematics (BCAM), Alameda Mazarredo 14, E-48009 Bilbao, Basque Country, Spain

[c] IKERBASQUE, Basque Foundation for Science, E-48011 Bilbao, Basque Country, Spain

ARTICLE INFO

Article history:
Received 2 August 2011
Received in revised form
3 January 2013
Accepted 3 January 2013
Available online 14 February 2013

Keywords:
Parabolic equations
Elliptic equations
Approximate controllability
Spike controls
Borel measures

ABSTRACT

We analyze the use of measures of the minimal norm to control elliptic and parabolic equations. We prove the sparsity of the optimal control. In the parabolic case, we prove that the solution of the optimization problem is a Borel measure supported in a set of Lebesgue measure zero. In both cases, the approximate controllability can be achieved efficiently by means of controls that are activated in some finite number of pointwise locations. We also analyze the corresponding dual problem.

1. Introduction

In this paper we address the issue of controlling elliptic and parabolic equations by means of sparse controls. As we shall see, when looking for the control of minimal measure the sparsity is ensured. This is in contrast with the fact that controls of the minimal L^2 norm end up being smooth and distributed everywhere on the support of the controller while controls of the minimal L^∞-norm are of bang–bang form (see [1]).

We first analyze the problem of approximate controllability for the heat equation. More precisely, we consider the parabolic equation

$$\begin{cases} y' - \Delta y = u & \text{in } Q = \Omega \times (0,T), \\ y = 0 & \text{on } \Sigma = \Gamma \times (0,T), \\ y(0) = y_0 & \text{in } \Omega, \end{cases}$$

where $y_0 \in L^2(\Omega)$ is fixed, $\Omega \subset \mathbb{R}^n$ is an open connected bounded set and Γ is the boundary of Ω, that we will assume to be Lipschitz.

We wish to choose the control u such that the associated state at time T, $y_u(T)$, is in the $L^2(\Omega)$-ball $\bar{B}_\varepsilon(y_d)$, where y_d represents the desired final state and $\epsilon > 0$ the admissible distance to the target.

It is well known that for any $\epsilon > 0$ it is possible to find $u \in L^2(Q)$ such that $y_u(T) \in \bar{B}_\varepsilon(y_d)$; see Lions [2]. In fact the same holds when the control u has its support in a subset ω of Ω of positive measure.

We are interested in building and analyzing the structure of the controls u of minimal energy. In the L^2-setting this can be done by considering the following minimization problem

$$\min_{y_u(T) \in \bar{B}_\varepsilon(y_d)} J(u) = \frac{1}{2}\|u\|^2_{L^2(Q)}.$$

It can be checked that this problem has a unique solution that is given by $\bar{u} = -\bar{\varphi}$, where $\bar{\varphi}$ is the unique solution of the adjoint equation

$$\begin{cases} -\bar{\varphi}' - \Delta \bar{\varphi} = 0 & \text{in } Q, \\ \bar{\varphi} = 0 & \text{on } \Sigma, \\ \bar{\varphi}(T) = \bar{g} & \text{in } \Omega, \end{cases}$$

for some $\bar{g} \in L^2(\Omega)$ satisfying

$$\int_\Omega \bar{g}(x)(y(x) - \bar{y}(x,T))dx \le 0 \quad \forall y \in \bar{B}_\varepsilon(y_d).$$

Above, $\bar{y}$ denotes the state associated to $\bar{u}$. This means that the control is smooth but active at almost every point of Ω and at all instant t. This makes these controls to be of little practical use in applications where one looks for controls with small support.

In some recent papers, the use of the L^1-norm instead of the L^2-norm was shown to be very efficient to obtain optimal controls with support in small regions of the domain, the domain being adjustable in terms of the tuning of suitable parameters entering in the cost functional; see [3–5], or [6].

* Corresponding author. Tel.: +34 942 201427; fax: +34 942 201829.
E-mail addresses: eduardo.casas@unican.es (E. Casas), zuazua@bcamath.org (E. Zuazua).

0167-6911/$ – see front matter
doi:10.1016/j.sysconle.2013.01.001

However, the above control problem has no solution, in general, if we replace the $L^2(Q)$-norm by the $L^1(Q)$-norm, i.e. if we take $J(u) = \|u\|_{L^1(Q)}$. To overcome this difficulty, the use of Borel measures in Q and the cost functional $J(u) = \|u\|_{M(Q)}$ was suggested in [7]. The supports of the optimal measures turn out to be small.

In fact, as we shall show, in the present context of the approximate controllability of the heat equation, the optimal measure $\bar{u}$ has a support of null Lebesgue measure. To be more precise, we will prove that given any time interval $[T_0, T_1]$, where $0 < T_0 < T_1 < T$, such that the controls are supported in $[T_0, T_1]$, then there exists a measure $\bar{u}_\Omega \in M(\Omega)$, having a support of null Lebesgue measure, such that setting $\bar{u} = \bar{u}_\Omega \otimes \delta_{T_1}$, the property $y_{\bar{u}}(T) \in \bar{B}_\varepsilon(y_d)$ holds. Here δ_{T_1} stands for the Dirac measure concentrated at $t = T_1$. This shows that, in particular, the optimal measure is concentrated on a set of points of Ω, with zero Lebesgue measure, at the final time instant T_1. We also prove the uniqueness of this optimal measure. These are the main contributions of this paper.

Hereafter, we will denote by $\bar{u}_\Omega \otimes \delta_{T_1}$, the measure in Q defined by

$$\langle \bar{u}_\Omega \otimes \delta_{T_1}, y\rangle = \int_\Omega y(x, T_1) d\bar{u}_\Omega(x).$$

As proved in [8], a measure in Ω can be efficiently approximated by a combination of Dirac measures. As a consequence, we deduce that the approximate controllability can be achieved by activating the controllers in some finite number of pointwise locations at the time T_1.

On the other hand, from the point of view of applications, it is natural to limit a priori the area where the controls can be placed. Thus, given a (possibly small) region $\omega \subset \Omega$, that we will assume to be an open non-empty set with finitely many connected components, we assume that the support of u is required to be in $\bar{\omega}$.

This leads to the study of the following optimal control problem

$$(\mathrm{P}) \begin{cases} \min J(u) = \|u\|_{M(Q_0)}, \\ (u, y_u(T)) \in M(Q_0) \times \bar{B}_\varepsilon(y_d), \end{cases}$$

where $Q_0 = (\Omega \cap \bar{\omega}) \times [T_0, T_1]$, $M(Q_0)$ being the space of real and regular Borel measures in Q_0 and y_u the solution of

$$\begin{cases} y' - \Delta y = u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = 0 & \text{in } \Omega. \end{cases} \tag{1}$$

Without loss of generality, we have taken the initial state $y_0 = 0$. Indeed, for $y_0 \neq 0$ we can consider the solution $\tilde{y}$ of the parabolic equation corresponding to $u = 0$ and change y_d by $y_d - \tilde{y}(T)$. Then the problem is formulated as above. In (1), u is extended by zero outside Q_0.

To avoid the trivial case where the optimal solution is $\bar{u} = 0$, hereafter we will assume that $\|y_d\|_{L^2(\Omega)} > \varepsilon$. On the other hand, let us observe that the choice $T_1 < T$ can be convenient not only for practical reasons, but it is theoretically necessary as well. Indeed, if T_1 is taken equal to T, then $y_u(T)$ does not belong, in general, to $L^2(\Omega)$. Therefore, the problem (P) is not well posed in the indicated spaces. We also take $T_0 > 0$ to avoid the use of (measure) controls as initial condition.

We will denote

$$C_0(Q_0) = \{y \in C(\bar{Q}_0) : y(x, t) = 0 \text{ on } \Sigma \cap (\partial\omega \times [T_0, T_1])\}.$$

Endowed with the maximum norm, this is a Banach space. Moreover, since Q_0 is a locally compact Hausdorff space, according to the Riesz representation theorem (see, for instance, Rudin [9, Theorem 6.19]), $M(Q_0)$ is identified with the dual of $C_0(Q_0)$ and

$$\|u\|_{M(Q_0)} = |u|(Q_0) = \sup_{y \in C_0(Q_0),\ \|y\|_\infty \le 1} \int_{Q_0} y(x, t)\, du(x, t),$$

where $|u|$ denotes the total variation measure associated to u.

The plan of the paper is as follows. In the next section, we analyze the control problem (P): we prove the existence and uniqueness of a solution, we get the optimality conditions and establish the spike structure of the optimal control. The dual problem (P*) is studied in Section 3. Some equivalent formulations for (P) and (P*) are considered in Section 4. As claimed in [7], convex duality is a powerful framework for solving non-smooth optimal control problems. This has motivated us to consider the study of the dual problems. Finally, the previous results are extended to the elliptic case in Section 5.

2. Analysis of the control problem (P)

Before analyzing the control problem (P), we will comment some known facts about the equation (1). First, we give a definition of solution of (1) and then we study the existence, uniqueness and continuity with respect to the measure u; see [10] or [11] for more details.

Definition 1. Given $p, r \in [1, 2)$, with $(2/r) + (n/p) > n + 1$, we will say that a function $y \in L^r([0, T], W_0^{1,p}(\Omega))$ is a solution of (1) if the following identity holds

$$\int_Q (-\phi' y + \nabla\phi\nabla y)\, dxdt = \int_{Q_0} \phi\, du \quad \forall \phi \in \Phi, \tag{2}$$

where

$$\Phi = \{\phi \in C^1(\bar{Q}) : \phi(x, T) = 0 \text{ in } \Omega \text{ and } \phi(x, t) = 0 \text{ on } \Sigma\}.$$

Theorem 1. *There exists a unique function $y \in L^r([0, T], W_0^{1,p}(\Omega))$ for all $p, r \in [1, 2)$, with $(2/r) + (n/p) > n + 1$, such that it is a solution of (1) and*

$$\int_Q (-\phi' - \Delta\phi) y\, dxdt = \int_{Q_0} \phi(x, t)\, du(x, t) \quad \forall \phi \in \Phi_\infty, \tag{3}$$

where $\Phi_\infty = \{\phi \in \Phi : \phi' + \Delta\phi \in L^\infty(Q)\}$. Moreover, there exists a constant $C_{r,p} > 0$ independent of u such that

$$\|y\|_{L^r([0,T],W_0^{1,p}(\Omega))} + \|y(T)\|_{L^2(\Omega)} \le C_{r,p}\|u\|_{M(Q_0)}. \tag{4}$$

Proof. Let us take a sequence of functions $\{u_k\}_{k=1}^\infty \subset C(\bar{Q})$ such that $u_k \rightharpoonup u$ weakly* in $M(Q)$. We can assume that $\mathrm{supp}(u_k) \subset \bar{Q}_\rho = \bar{\Omega} \times [0, T_1 + \rho]$, with $T_1 + \rho < T$, and $\|u_k\|_{L^1(Q)} \le \|u\|_{M(Q_0)}$. The standard way to get this sequence is making the convolution of u with a sequence of mollifiers. Now, we take $y_k \in L^2([0, T], H_0^1(\Omega)) \cap C([0, T], L^2(\Omega))$ the solution of

$$\begin{cases} y_k' - \Delta y_k = u_k & \text{in } Q, \\ y_k = 0 & \text{on } \Sigma, \\ y_k(0) = 0 & \text{in } \Omega. \end{cases} \tag{5}$$

Then, we can argue as in [10, Theorem 6.3] to deduce that

$$\|y_k\|_{L^r([0,T],W_0^{1,p}(\Omega))} \le C_{r,p}\|u\|_{M(Q_0)}$$

for some constant independent of k. As in [10], we get for a subsequence that $y_k \rightharpoonup y$ in $L^r([0, T], W_0^{1,p}(\Omega))$, which is the unique solution of (1) satisfying (3). To get the estimate for $y(T)$ in $L^2(\Omega)$ we proceed as follows. Given $g \in L^2(\Omega)$, take $\varphi_g \in L^2([0, T], H_0^1(\Omega)) \cap C([0, T], L^2(\Omega))$ the solution of

$$\begin{cases} \phi' + \Delta\phi = 0 & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \phi(T) = g & \text{in } \Omega. \end{cases} \tag{6}$$