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Input-to-state stability of nonlinear systems based on an indefinite Lyapunov function

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ABSTRACT

This study investigated the input-to-state stability (ISS) and integral ISS (iISS) of nonlinear systems and discovered new sufficient conditions for them. These conditions are more relaxed than others in that they employ an indefinite Lyapunov function rather than a negative definite one. Thus, a nonlinear time-varying system satisfying them has uniform asymptotic stability. Two numerical examples show their effectiveness and advantages.

1. Introduction

The concept of input-to-state stability (ISS) [1] ensures asymptotic stability for zero input systems. It is important in the analysis and synthesis of nonlinear systems. [1] proposed sufficient conditions for ISS in which the derivative of a Lyapunov function had to be negative. Later, [2] provided useful characterizations of ISS. Moreover, [3,4] demonstrated the equivalence of ISS and uniform asymptotic gain.

On the other hand, [5] first proposed the idea of integral input-to-state stability (iISS), which is a nonlinear generalization of $\mathcal{L}^2$ stability. It turned out to be strictly weaker than ISS. We now know that a time-invariant system is iISS if there exists a positive definite Lyapunov function whose derivative along the system is negative definite [6].

Generally speaking, the problem of achieving ISS is more difficult for a time-varying system than for a time-invariant one because a Lyapunov function for the former is also a function of time. [7] considered this and devised a useful method of transforming a weak Lyapunov function with a negative semi-definite derivative into a Lyapunov function with a negative definite derivative. [8] used the same method to demonstrate the ISS of a time-varying system with a weak Lyapunov function. Furthermore, [9] discussed the non-uniform ISS of a time-varying system.

[1–6,8,9] employed a Lyapunov function to demonstrate the ISS of both time-invariant and time-varying systems. However, the requirement that the Lyapunov function have a negative definite derivative is very strict. Recently, [10] showed that the the diffusion operator associated with the stochastic functional differential equations with Markovian switching of the Lyapunov function along a solution of the system does not always have to be negative. That motivated us to examine the ISS and iISS of nonlinear systems.

This paper first presents a new comparison principle for estimating an upper bound on the state of a system for which the derivative of its Lyapunov function may be indefinite, rather than negative definite, as it must be in other studies. This greatly extends previous work in this field. It leads to a new criterion for the ISS of nonlinear time-varying systems through the construction

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of a $\mathcal{KL}$-function and the use of a new type of Lyapunov function for which the derivative is allowed to be positive definite during some periods. We call this type of function an *indefinite Lyapunov function*. As a by-product, a new criterion for uniform asymptotic stability for nonlinear systems with zero input is derived. Then, two theorems for the iISS of nonlinear systems are established based on an indefinite Lyapunov function. Finally, two numerical examples illustrate the effectiveness and advantages of these concepts.

This paper uses the following notation: $\mathbb{R}$ is the set of real numbers. $\mathbb{R}^+$ is the set of all nonnegative real numbers. $a \wedge b$ and $a \vee b$ are the minimum and maximum of a and b, respectively. $|v|$ is the Euclidean norm of the real vector, v. $\mathcal{L}_\infty$ is the space of measurable and locally essentially bounded functions. $\|\cdot\|$ is the essential supremum norm of a function. $u_{[t_0,t]}(s)$ is the truncation of $u(s)$; that is, $u_{[t_0,t]}(s) = u(s)$ when $t_0 \le s \le t$ and $u_{[t_0,t]}(s) = 0$ for $s > t$. The function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{K}$-function if γ is continuous and strictly increasing for $\gamma(0) = 0$; it is denoted by $\gamma \in \mathcal{K}$. γ is a $\mathcal{K}_\infty$-function if it is a $\mathcal{K}$-function and also satisfies $\lim_{t\to\infty} \gamma(t) = \infty$; it is denoted by $\gamma \in \mathcal{K}_\infty$. The function $\sigma(s,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a $\mathcal{KL}$-function if it is a $\mathcal{K}$-function for a fixed t and the mapping $\sigma(s,t)$ decreases to zero as $t \to \infty$ for a fixed s; it is denoted by $\sigma \in \mathcal{KL}$. "ess sup" is the essential supremum of an essentially bounded function. Finally, $[x]$ indicates the round function of x.

The rest of the paper is organized as follows: Section 2 first defines ISS and iISS, and then presents the new characterizations of ISS and iISS based on an indefinite Lyapunov function. It also discusses uniform asymptotic stability. Section 3 gives two numerical examples that illustrate the effectiveness and the advantages of the new characterizations. Section 4 makes some concluding remarks.

2. Main propositions

First, we define ISS and iISS. Consider the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{1}$$

where $x(t)$ is the state; $\mathbb{R} \to \mathbb{R}^n$; $u(t)$ is the control input; $\mathbb{R}^+ \to \mathbb{R}^m$, which is assumed to be measurable and locally essentially bounded; and $f : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be locally Lipschitz in (t, x), to be uniformly continuous in u, and to satisfy $f(t, 0, 0) = 0$.

Carathèodory conditions ensure that there exists a unique maximal solution $x(t, x(t_0), u)$ for System (1) over $[t_0, d)$ for an initial value $x(t_0) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and an initial time $t_0 \ge 0$ [11,12]. Note that $d \in (t_0, \infty]$.

The definitions of ISS [1] and iISS [6] are now reported.

Definition 1 (*ISS*)**.** System (1) is said to be input-to-state stable (ISS) if there exist a $\mathcal{KL}$-function $\sigma(s,t)$ and a $\mathcal{K}$-function $\gamma(s)$ such that, for any initial state $x(t_0)$, any measurable, locally essentially bounded input $u(t)$, the solution exists for all $t \ge t_0$ and satisfies

$$|x(t)| \le \sigma(|x(t_0)|, t - t_0) + \gamma(\|u_{[t_0,t]}\|). \tag{2}$$

Definition 2 (*iISS*)**.** System (1) is said to be iISS if there exist a $\mathcal{KL}$-function σ, a $\mathcal{K}_\infty$-function α, and a $\mathcal{K}$-function γ, such that, for any initial state $x(t_0)$, any measurable, locally essentially bounded input $u(t)$, the solution exists for all $t \ge t_0$ and satisfies

$$\alpha(|x(t)|) \le \sigma(|x(t_0)|,\ t - t_0) + \int_{t_0}^{t} \gamma(|u(\tau)|)d\tau. \tag{3}$$

Now, we are ready to give the main propositions.

Consider the measurable, locally essentially bounded property of an input function. The following propositions hold almost all the time.

Lemma 1. *Suppose that $y : \mathbb{R}^+ \to \mathbb{R}^+$ is an absolutely continuous function; $u : \mathbb{R}^+ \to \mathbb{R}^m$ is a measurable, locally essentially bounded mapping; $\phi : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function; and $\rho \in \mathcal{K}$. If, for almost all $t \ge t_0$,*

$$\dot{y}(t) \le \phi(t)y(t), \quad \forall y(t) \ge \rho(|u(t)|)$$

with an initial value $y(t_0) \ge 0$, then the following estimate holds for all $t \ge t_0$:

$$y(t) \le y(t_0)e^{\int_{t_0}^{t} \phi(\tau)d\tau} + \operatorname*{ess\,sup}_{t_0 \le s \le t} \rho(|u(s)|)e^{\int_{t_0}^{t} \phi^+(\tau)d\tau}, \tag{4}$$

where $\phi^+(\tau) = \phi(\tau) \vee 0$.

Proof. First, for the inequality

$$y(s) \ge \rho(|u(s)|), \tag{5}$$

we consider two cases:

(a) (5) holds for almost all $s:\ t_0 \le s \le t$; and
(b) (5) does not hold for almost all $s:\ t_0 \le s \le t$.

For Case (a), Gronwall's inequality gives us

$$y(t) \le y(t_0)e^{\int_{t_0}^{t} \phi(\tau)d\tau}. \tag{6}$$

For Case (b), we know that a measure of the set $\{t_0 \le s \le t : y(s) \le \rho(|u(s)|)\}$ is greater than zero. Let

$$t^* = \operatorname{ess\,sup}\{t_0 \le s \le t : y(s) \le \rho(|u(s)|)\}.$$

Then, either $t^* < t$ or $t^* = t$. If $t^* < t$, then for almost all $t^* < s \le t$,

$$\dot{y}(s) \le \phi(s)y(s).$$

So, from Gronwall's inequality, we have

$$y(t) \le y(s)e^{\int_{s}^{t} \phi(\tau)d\tau} \tag{7}$$

for all $t^* < s \le t$. The continuity of y and ϕ combined with the fact $\phi(t) \le \phi^+(t)$ yield

$$y(t) \le y(t^*)e^{\int_{t^*}^{t} \phi(\tau)d\tau} \le y(t^*)e^{\int_{t^*}^{t} \phi^+(\tau)d\tau}.$$

From the definition of t^*, the above inequality gives

$$y(t) \le \operatorname*{ess\,sup}_{t_0 \le s \le t} \rho(|u(s)|)e^{\int_{t_0}^{t} \phi^+(\tau)d\tau}. \tag{8}$$

If $t^* = t$, then there exists a constant δ such that, for all $t - \delta < s < t$,

$$y(s) \le \rho(|u(s)|) \le \operatorname*{ess\,sup}_{t_0 \le s_1 \le s} \rho(|u(s_1)|).$$

If we let $s \to t$, it follows from the continuity of y and the monotonicity of $\operatorname{ess\,sup}_{t_0 \le s_1 \le s} \rho(|u(s_1)|)$ that

$$y(t) \le \operatorname*{ess\,sup}_{t_0 \le s \le t} \rho(|u(s)|)e^{\int_{t_0}^{t} \phi^+(\tau)d\tau} \tag{9}$$

holds for all $t > t_0$.

Now, combining (6), (8), and (9), we conclude that

$$y(t) \le y(t_0)e^{\int_{t_0}^{t} \phi(\tau)d\tau} + \operatorname*{ess\,sup}_{t_0 \le s \le t} \rho(|u(s)|)e^{\int_{t_0}^{t} \phi^+(\tau)d\tau}.$$

This completes the proof. □