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Recursive estimators with Markovian jumps[☆]

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ABSTRACT

Recursive stochastic algorithms have various applications. In the literature, it is assumed that the true value lies in a connected domain. But, in many cases, it is known that the true value is contained in the union of a finite number of pairwise disjoint sets instead of a connected domain. In these situations, the existing algorithms may be not applicable. To cope with this problem, this paper proposes recursive stochastic algorithms with (event-triggered) Markovian jumps and presents sufficient conditions for almost sure convergence of the proposed algorithms. As an example of applications, this paper significantly improves an existing adaptive algorithm with the proposed method for consistent estimation of non-minimum phase zeros.

1. Introduction

The study of recursive stochastic algorithms, also known as stochastic approximations, was initiated by Robbins and Monro [1], who developed and analyzed a recursive procedure for finding the root of a real-valued function of a real variable. The use of recursive stochastic algorithms is now very widespread across varied applications such as system identification, adaptive control, transmission systems, machine learning, adaptive filtering for signal processing, and several aspects of pattern recognition (see [2–7] and the references therein). In the literature ([2,3,8–12,4,5,13–16,6] and the references therein), it is usually assumed that the true value lies in a connected compact domain, and the recursive scheme starts with a chosen interior point of this domain. To keep the estimator in the domain, a projection or a resetting mechanism is employed in the recursive algorithm itself (see, e.g., [17,11,12,4,13,14,6]). However, in many practical cases, it is known that the true value is contained in the union of a finite number of pairwise disjoint compact subsets instead of a connected compact domain. It could be possible to find a larger connected compact domain that includes the union of the subsets and then implement the recursive algorithm over the larger connected domain. But sometimes this is unfeasible, since, for example, there are some singular points, and hence the algorithm is not well defined on such a larger domain (see Section 4). To deal with this problem, we introduce (event-triggered) Markovian jumps into the recursive stochastic algorithms (as a resetting mechanism) and present the sufficient conditions for convergence with probability one of our proposed algorithms.

Our proposed method may have many promising applications (see, e.g., [18–20] and the references therein). One of these is the consistent estimation of NMP (non-minimum phase) zeros. NMP zeros play important roles in many control applications since they limit closed loop performance. Recently, identification of real-valued NMP zeros of discrete-time LTI (linear time-invariant) systems has been studied in a few works (see, e.g., [21–23]). Among the key results, [23] proposes an adaptive algorithm for consistent estimation of real-valued NMP zeros in stable LTI systems by applying a result in [11] (see Theorem 4.1 in [11] and also Appendix A in [23]), which shows that it is possible to estimate a real-valued NMP zero with multiplicity one outside the unit circle consistently using a simple two-parameter FIR (finite impulse response) model if the input can be manipulated and some prior information is available. In [23], it is assumed that not only is some prior knowledge about the location of the NMP zero of interest known, but also the sign of the first impulse response coefficient is known. Recently, [24] modified this adaptive algorithm by introducing a numerical transform and removing one important condition on the prior knowledge, namely, that of the sign of the first impulse response coefficient. However, prior information on the sign of the zero of interest is still required and, moreover, the modified method is only applicable to the estimation of the farthest ones. Without such prior system knowledge, these results are not

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applicable (see [24]). But, in many practical cases, the sign of the NMP zero or that of the first impulse response coefficient cannot be known in advance. As an example of applications, we improve the recursive algorithm presented in [23] with our proposed method so that it works for consistent estimation of zeros in stable LTI systems when prior knowledge on either the sign of the zero of interest or that of the high frequency gain is unavailable.

2. Recursive algorithms with Markovian jumps

Our problem will be embedded in an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{E}[\cdot]$ be the expectation operator with respect to the probability measure.

A general form of the Robbins–Monro algorithms is given by (see [2,3,5])

$$x_{n+1} = x_n + \gamma_{n+1} H(x_n, \Phi_{n+1}), \tag{1}$$

where $\{x_n\}_{n\geq 0}$ is the sequence of vectors to be recursively updated; $\{\Phi_n\}_{n\geq 1}$ is a sequence of random vectors representing the on-line observations of the system in the form of a state vector (see (4) below); $\{\gamma_n\}_{n\geq 1}$ is a sequence of scalar gains satisfying

$$\begin{cases} \gamma_n \geq 0, \quad \forall n \geq 1 \\ \displaystyle\sum_{n=1}^{\infty} \gamma_n = \infty, \\ \displaystyle\sum_{n=1}^{\infty} n\gamma_n^{\alpha} < \infty \quad \text{for some } \alpha > 1. \end{cases} \tag{2}$$

$H(x, \Phi)$ is the function that essentially defines how the parameter x is updated as a function of new observations and may admit discontinuities, while

$$h(x) \triangleq \lim_{n\to\infty} \mathbb{E}_x[H(x, \Phi_n)] \tag{3}$$

is well defined on a finite number of compact sets $D_1, D_2, \ldots, D_M$, where $D_m, m \in \mathcal{S}_M = \{1, 2, \ldots, M\}$, is connected and $D_j \cap D_m = \emptyset$ if $j \neq m$. The initial estimate $\xi_m \in \operatorname{int} D_m$ can be chosen with extra prior knowledge or, otherwise, as any interior point of $D_m, m \in \mathcal{S}_M$.

Assume that $\{\Phi_n\}$ is conditionally linear (see [13,14]); that is,

$$\Phi_{n+1} = A(x_n)\Phi_n + B(x_n)\eta_n, \tag{4}$$

where $A(x)$ and $B(x)$ are matrices with bounded entries for all $x \in D_m, m \in \mathcal{S}_M$; noise $\{\eta_n\}$ is a sequence of independent random variables such that

$$\sup_n \mathbb{E}\left[e^{\bar{\varepsilon}|\eta_n|^2} \right] < \infty \tag{5}$$

holds for some $\bar{\varepsilon} > 0$, which is certainly satisfied for wide-sense stationary Gaussian sequences (see [8,11,12]).

Moreover, assume that the time-varying system (4) is BIBO (bounded input–bounded output) stable. According to Theorem 2.1 in [25], the switching system (4) is BIBO stable if and only if it is uniformly exponentially stable, or, equivalently, uniformly asymptotically stable (see, e.g., [26]). Clearly, the BIBO stability of (4) is guaranteed by the joint stability of $A(x)$ for all $x \in D_m, m \in \mathcal{S}_M$; that is (see, e.g., Condition 4.1 in [11] and Condition 3.7 in [12]), there exist a symmetric positive definite matrix V and a constant $\lambda \in (0, 1)$ such that

$$A^T(x)VA(x) \leq \lambda V \tag{6}$$

for all $x \in D_m, m \in \mathcal{S}_M$. But the BIBO stability of system (4) is also ensured when the bounded set of matrices $\Sigma_A = \{A(x) : x \in D_m, m \in \mathcal{S}_M\}$ is LCP (left convergent products), i.e., every left-infinite product $\lim_{n\to\infty} A_n \cdots A_2 A_1$ converges, where $A_k \in \Sigma_A$ for all $k = 1, 2, \ldots, n$ (see [27,28]). In the example for applications of our proposed method, we will show the BIBO stability of system (4) by applying an important result of the joint spectral radius (see [29,27,28] and the references therein) in Section 4.

Let us present the following recursive stochastic algorithm with event-triggered Markovian jumps:

$$\begin{aligned} &x_{n+1-} = x_n + \gamma_{n+1} H(x_n, \Phi_{n+1}), \\ &x_0 = \xi_{j_0} \in \operatorname{int} D_{j_0}, \quad j_0 \in \mathcal{S}_M \end{aligned} \tag{7}$$

$$x_{n+1} = \begin{cases} x_{n+1-}, & x_n \in D_j, x_{n+1-} \in D_j, j \in \mathcal{S}_M \\ \tilde{\xi}, & \text{otherwise,} \end{cases} \tag{8}$$

where $x_{n+1} = \tilde{\xi}$ will be randomly reset to $\xi_m \in \operatorname{int} D_m, m \in \mathcal{S}_M$, according to the Markov transition matrix

$$P_M = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1M} \\ p_{21} & p_{22} & \cdots & p_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ p_{M1} & p_{M2} & \cdots & p_{MM} \end{bmatrix} \tag{9}$$

with $p_{jm} > 0$ and $\sum_{m=1}^{M} p_{jm} = 1$ for all $1 \leq j, m \leq M$; that is, $\mathbb{P}\{x_{n+1} = \xi_m\} = p_{jm}$ if $x_n \in D_j$ and $x_{n+1-} \notin D_j$. The entries of the Markov transition matrix (9) can be assigned values with prior knowledge. In the absence of other information, a reasonable choice is to take $p_{jm} = 1/M$ for all $1 \leq j, m \leq M$, which is employed in the application example of our proposed method (see Section 4).

3. Convergence analysis: the ODE (ordinary differential equation) method

The ODE associated with the algorithm is now introduced as follows:

$$\frac{dy(v)}{dv} = h(y(v)) \tag{10}$$

for $v \geq 0$ with initial value $y(0) = y_0 = x_1$, where function $h(\cdot)$ is given by (3). The solution of ODE (10) is denoted by $y(v) = y(v; y_0)$ for $v \geq 0$. Assume that there is an attractor x_m^* of ODE (10) such that the distance between x_m^* and D_m is bigger than 0 (i.e., $d(x_m^*, D_m) > 0$) and $D_m \subset D_m^*$ for $m = 1, 2, \ldots, M - 1$, while ODE (10) has an asymptotically stable equilibrium point $x_* \in \operatorname{int} D_M$ with $D_M \subset D_*$, where D_m^* and D_* are the domains of attraction of x_m^* and x_*, respectively. Moreover, $Y_M \subset \operatorname{int} D_M$, where

$$Y_M = \{y(v) : v \geq v_0, y(v_0) = \xi_M\}. \tag{11}$$

Let us present the following convergence result.

Theorem 3.1. *The discrete-time process $\{x_n\}$ computed by the recursive stochastic algorithm with Markovian jumps (7)–(9) converges to x_* a.s. (almost surely) as $n \to \infty$.*

Proof. Without loss of generality, assume that $x_0 = \xi_{j_0} \in \operatorname{int} D_{j_0}$ and $y_0 = x_1 \in D_m$ with $1 \leq j_0, m \leq M - 1$. The proof is rather technical, so it is divided into five steps, which will show the following: the state vector $\{\Phi_n\}_{0\leq n\leq \bar{N}}$ is bounded a.s. for any $\bar{N} \geq 1$ in step 1; the estimator $\{x_n\}$ hits the boundary of $D_m, m = 1, 2, \ldots, M - 1$, and then enters D_M in finite time a.s. in step 2 and step 3, respectively; there exists a finite random variable $\rho_k < \infty$ such that the estimator $\{x_n\}$ stays in D_M for all $n \geq \rho_k$ with probability one in step 4; and, in step 5, the estimator $\{x_n\}$ converges to x_* a.s. as $n \to \infty$. For $n > 1$, we say that x_n hits the boundary of D_m, denoted by ∂D_m, if $x_{n-1} \in D_m$ and $x_{n-} \notin D_m$.

Step 1: For system (4), we have

$$\mathbb{P}\left\{ \sup_{0\leq n\leq \bar{N}} |\Phi_n| < \infty \right\} = 1 \tag{12}$$