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New results on sampled-data feedback stabilization for autonomous nonlinear systems

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ABSTRACT

Sufficient conditions are established for sampled-data feedback global asymptotic stabilization for nonlinear autonomous systems. One of our main results is an extension of the well known Artstein–Sontag theorem on feedback stabilization concerning affine in the control systems. A second aim of the present work is to provide sufficient conditions for sampled-data feedback asymptotic stabilization for two interconnected nonlinear systems. Lie algebraic sufficient conditions are derived for the case of affine in the control interconnected systems without drift terms.

1. Introduction

In the recent literature on nonlinear control theory an important area that has received much attention is the stabilization problem by means of sampled-data feedback; see for instance, [1–18] and relative references therein, where sufficient conditions are established for the existence of sampled-data and hybrid feedback controllers exhibiting stabilization. We also mention the recent contributions [19,20,12,21], where, under the presence of Control Lyapunov Functions (CLF), "triggering" techniques are developed for the determination of the set of the sampling time instants for the corresponding sampled-data controller. Particularly, in [20] a "minimum attention control" approach is adopted, exhibiting minimization of the open loop operation of the sampled-data control. In [12] a universal formula is proposed for the sample-data feedback exhibiting "event-based" stabilization of affine in the control systems. The corresponding result constitutes a generalization of Sontag's well known result in [22].

In the recent author's works [23–25] the concept of *Weak Global Asymptotic Stabilization by Sampled-Data Feedback* (SDF-WGAS) is introduced and Lyapunov-like sufficient conditions for the existence of a sampled-data feedback stabilizer have been established. These conditions are weaker than those proposed in earlier contributions on the same subject. The present paper constitutes a continuation of the previously mentioned author's works. The paper is organized as follows:

The current section contains the precise definition of the SDF-WGAS, as originally given in [24]. In Section 2 we establish a general result (Proposition 1), which provides Lyapunov characterizations of certain properties relying on asymptotic controllability for general nonlinear systems:

$$\dot{x} = F(x, u), \quad (x, u) \in \mathbb{R}^n \times \mathbb{R}^\ell \tag{1.1a}$$

$$F(0, 0) = 0 \tag{1.1b}$$

where we assume that $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz continuous.

Sections 3 and 4 are devoted to applications of Proposition 1 for the solvability of the SDF-WGAS problem for a certain class of nonlinear systems. In Section 3 we apply the result of Proposition 1 to derive an extension of the so called Artstein–Sontag theorem (see [26,22] and also [2,3] for recent extensions). Particularly, in Proposition 2 a sufficient condition for the solvability of the SDF-WGAS problem for the case of affine in the control systems:

$$\dot{x} = f(x) + ug(x), \quad (x, u) \in \mathbb{R}^n \times \mathbb{R} \tag{1.2a}$$

$$f(0) = 0 \tag{1.2b}$$

is established. This condition is weaker than the familiar hypothesis imposed in [26] and includes the Lie bracket between the vector fields f and g. In Section 4 we use the result of Proposition 1 to provide a small-gain criterion for the possibility of sampled-data feedback global stabilization for composite systems of the form:

$$\begin{aligned} \Sigma_1 &: \dot{x} = f(x, y, u) \\ \Sigma_2 &: \dot{y} = g(x, y, u) \end{aligned} \tag{1.3a}$$

$$\begin{aligned} &(x, y, u) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^\ell \\ &f(0, 0, 0) = 0, \qquad g(0, 0, 0) = 0. \end{aligned} \tag{1.3b}$$

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The corresponding result (Proposition 3) extends the main result in [25] and partially extends the well known results in the literature (see for instance [27,28]) establishing small-gain criteria for composite systems Σ_1, Σ_2 with no controls and particularly those in [29–31] (see also relative references therein), where each subsystem does not necessarily satisfy the Input-to-State-Stability (ISS) property. As a consequence of Proposition 3, we provide a partial extension of the well known result in [32] due to Coron, concerning the solvability of the stabilization problem by means of smooth time-varying feedback for the affine in the control systems without drift term. In the present work we consider nonholonomic composite systems of the following form:

$$\dot{\xi} = F(\xi, u) := \sum_{i=1}^{\ell} u_i F_i(\xi) \tag{1.4a}$$

$$F_i(\xi) := \begin{pmatrix} A_i(\xi) \\ B_i(\xi) \end{pmatrix}, \quad i = 1, \dots, \ell, \; \xi := (x, y) \in \mathbb{R}^n \times \mathbb{R}^m \tag{1.4b}$$

$$A_i(0) = 0, \qquad B_i(0) = 0, \quad i = 1, \dots, \ell \tag{1.4c}$$

and in Proposition 4 we derive a set of Lie algebraic sufficient conditions guaranteeing SDF-WGAS for the case (1.4), being weaker than the *accessibility rank condition* imposed in [32].

Notation and definitions: Throughout the paper, we adopt the following notation. By x^T we denote the transpose of a given vector $x \in \mathbb{R}^n$. By K we denote the set containing all continuous strictly increasing functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ and K_∞ denotes the subset of K that is constituted by all $\phi \in K$ with $\phi(t) \to \infty$ as $t \to \infty$. We denote by $\pi(\cdot, s, x_0, u)$ the trajectory of (1.1a) with $\pi(s, s, x_0, u) = x_0$ corresponding to certain (measurable and essentially bounded) control $u : [s, T_{\max}) \to \mathbb{R}^m$, where $T_{\max}$ is the corresponding maximal existing time of the trajectory.

Definition 1. We say that (1.1) is *Globally Asymptotically Controllable* at zero (GAC), if for any $x_0 \in \mathbb{R}^n$ there exists a control $u_{x_0}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^\ell$ such that $\pi(t, 0, x_0; u_{x_0})$ exists for all $t \geq 0$ and the following properties hold:

$$\text{Stability}: \; \forall \varepsilon > 0 \Rightarrow \exists \delta = \delta(\varepsilon) > 0 : |x_0| \leq \delta \Rightarrow \left|\pi(t, 0, x_0, u_{x_0})\right| \leq \varepsilon, \quad \forall t \geq 0 \tag{1.5a}$$

$$\text{Attractivity}: \; \lim_{t \to \infty} \pi(t, 0, x_0, u_{x_0}) = 0, \quad \forall x_0 \in \mathbb{R}^n. \tag{1.5b}$$

Definition 2. We say that (1.1) is *Weakly Globally Asymptotically Stabilizable by Sampled-Data Feedback* (SDF-WGAS), if for any constant $\sigma > 0$ there exist a map $T : \mathbb{R}^n \to \mathbb{R}^+$ satisfying $T(0) = 0$ and

$$T(x) \leq \sigma, \quad \forall x \in \mathbb{R}^n \setminus \{0\} \tag{1.6}$$

and a map $\varphi := \varphi(t, s, x) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^\ell$ with $\varphi(\cdot, \cdot, 0) = 0$, being measurable and essentially bounded with respect to the first two variables, such that for every $x_0 \neq 0$, a sequence of times

$$t_0 := 0 < t_1 < t_2 < \cdots < t_\nu < \cdots; \quad t_\nu \to \infty \tag{1.7}$$

can be found, in such a way that, if we denote:

$$u_{x_0}(t) := \varphi(t, t_i, \pi(t_i)), \quad t \in [t_i, t_{i+1}) \; i = 0, 1, 2, \dots \tag{1.8a}$$

$$\pi(t_i) := \pi(t_i, t_{i-1}, \pi(t_{i-1}), u_{x_0}), \quad i = 1, 2, \dots; \; \pi(t_0) := x_0 \tag{1.8b}$$

then

$$t_{i+1} - t_i = T(\pi(t_i)), \quad i = 0, 1, 2, \dots \tag{1.9}$$

and (1.1) is GAC by means of the controller $u_{x_0}(\cdot)$ as defined in (1.8); equivalently, the system

$$\dot{x}(t) = f(x(t), u_{x_0}(t)), \quad t \in [t_i, t_{i+1}) \text{ with } u_{x_0}(\cdot) \text{ as defined in (1.8)} \tag{1.10}$$

satisfies properties (1.5a) and (1.5b).

2. A general result

Consider the system (1.1) and assume that there exist a closed set $D_1 \subset \mathbb{R}^n$ containing zero $0 \in \mathbb{R}^n$, an open neighborhood D_2 of D_1, constants

$$0 \leq L_1 < L_2 \leq +\infty, \tag{2.1}$$

a continuous function $\Phi : \mathbb{R}^n \to \mathbb{R}^+$ with

$$\sup\{\Phi(x) : x \in D_1\} \leq L_1; \tag{2.2a}$$

$$0 < \Phi(x) < L_2, \quad \forall x \in D_2 \setminus D_1; \tag{2.2b}$$

$$\Phi(x) \geq L_2, \quad \forall x \in \mathbb{R}^n \setminus D_2, \text{ provided that } \mathbb{R}^n \setminus D_2 \neq \varnothing; \tag{2.2c}$$

$$\Phi(x) = L_2, \forall x \in \partial D_2 \text{ and for every sequence } \{x_\nu \in D_2 \setminus D_1\} \text{ with } \lim_{\nu \to \infty} |x_\nu| = \infty \text{ it holds } \limsup_{\nu \to \infty} \Phi(x_\nu) = L_2 \tag{2.2d}$$

and a function $\beta \in K$, such that for every constant $\sigma > 0$ and for every $x \in D_2 \setminus D_1$ there exists a time $\tau = \tau(x) \in (0, \sigma]$ and a control $u_x(\cdot) : [0, \tau] \to \mathbb{R}^m$ satisfying the following properties:

$$\Phi(\pi(\tau, 0, x, u_x)) < \Phi(x), \tag{2.3a}$$

$$\Phi(\pi(t, 0, x, u_x)) \leq \beta(\Phi(x)), \quad \forall t \in [0, \tau]. \tag{2.3b}$$

The first two statements of the following proposition generalize the result in [24, Proposition 1] and are used in Section 4 for the derivation of sufficient conditions for SDF-WGAS for the cases (1.3) and (1.4).

Proposition 1. *For system (1.1) assume that (2.1)–(2.3) are fulfilled. Then the following hold:*

(i) *For any constant $\sigma > 0$ there exists a map $T : D_2 \setminus D_1 \to \mathbb{R}^+ \setminus \{0\}$ satisfying (1.6) and a map $\varphi := \varphi(t, s, x) : \mathbb{R}^+ \times \mathbb{R}^+ \times (D_2 \setminus D_1) \to \mathbb{R}^\ell$ with the same regularity properties as those in Definition 2, such that for every $x_0 \in D_2 \setminus D_1$ there exists an increasing sequence of times $\{t_\nu\}$, in such a way that, if we denote $u_{x_0}(t) := \varphi(t, t_i, \pi(t_i))$, $t \in [t_i, t_{i+1})$, $i = 0, 1, 2, \dots, \pi(t_i) := \pi(t_i, t_{i-1}, \pi(t_{i-1}), u_{x_0})$, $i = 1, 2, \dots$; $\pi(t_0) := x_0$, then (1.9) holds and, if we consider the resulting system (1.10), then for every neighborhood $N \subset D_2$ of D_1 there exists a time $\tau := \tau(x_0) > 0$ such that its trajectory initiated from $x_0 \in D_2 \setminus N$ satisfies*

$$\pi(\tau, 0, x_0, u_{x_0}) \in \operatorname{int} N. \tag{2.4}$$

(ii) *If, in addition to previous assumptions, we assume that*

$$D_1 = \{0\}, \qquad L_1 = 0 \tag{2.5}$$

then for the trajectories of (1.10) both properties (1.5a), (1.5b) hold, provided that $x_0 \in D_2$.

(iii) *If, in addition to (2.5), we assume that*

$$D_2 = \mathbb{R}^n, \qquad L_2 = \infty \tag{2.6}$$

then system (1.1) is SDF-WGAS.

Remark 1. (i) The last statement of Proposition 1 coincides with the main result obtained in [24] and constitutes a generalization of Theorem 17 in [33]. To be precise, Proposition 1 in [24] asserts that system (1.1) is SDF-WGAS, if there exist a continuous, positive definite and proper function $\Phi : \mathbb{R}^n \to \mathbb{R}^+$ and a function $\beta \in K$ such that, for every constant $\sigma > 0$ and for every $x \in \mathbb{R}^n \setminus \{0\}$, a