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Global exponential observers for two classes of nonlinear systems

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ABSTRACT

This paper develops sufficient conditions for the existence of global exponential observers for two classes of nonlinear systems: (i) the class of systems with a globally asymptotically stable compact set, and (ii) the class of systems that evolve on an open set. In the first class, the derived continuous-time observer also leads to the construction of a robust global sampled-data exponential observer, under additional conditions. Two illustrative examples of applications of the general results are presented: one is a system with monotone nonlinearities and the other is a chemostat system.

1. Introduction

One of the biggest challenges of mathematical control theory has been the problem of constructing state observers for nonlinear systems. This problem has attracted a lot of attention in the literature in the past decades; it has been approached with a variety of methods and from a variety of points of view (see for instance [1–12] and references therein). In this work, we focus on nonlinear forward complete systems of the form

$$\dot{x} = f(x, u), \quad x \in \Re^n, u \in U, \tag{1.1}$$

where $U \subseteq \Re^m$ is a non-empty set, $f: \Re^n \to \Re^n$ is a smooth vector field, and the output is given by

$$y = h(x), \tag{1.2}$$

where $h: \Re^n \to \Re^k$ is a smooth mapping. The aim is to construct global exponential observers, i.e., observers with guaranteed exponential rate of convergence of the estimation error.

Available methods for global exponential observers include high-gain observers for globally Lipschitz systems [6] as well as circle-criterion observers, primarily for nonlinear systems with monotone nonlinearities [1,5]. In transformation-based observers, originally developed in local form in [8] and subsequently in [9,10], and in global form in [2], the system is mapped to a linear system, and the design of the observer is performed in transformed coordinates, where exponential convergence is imposed. Finally, global dead-beat observers are designed in [7] for a class of systems linear in the unmeasured state components (global dead-beat observers are by definition global exponential observers).

In this work, we present sufficient conditions for the existence of exponential observers for two important classes of nonlinear systems, which are not covered by the above methods.

(1) Nonlinear systems with an asymptotically stable compact set.
(2) Nonlinear systems evolving on open sets.

For both classes of systems, the proposed construction of the global exponential observer starts with a "candidate observer", which is subsequently modified by adding a correction term, in order to satisfy appropriate Lyapunov inequalities. It should be emphasized that explicit formulae for the observers are provided in each case, and therefore the control practitioner can directly apply the results of the paper.

In Section 2, where we study the first class of systems, the "candidate observer" is a local observer over a certain compact set, whereas the correction term forces the trajectory to enter the compact set in finite time. The derived continuous-time observer can also lead to the construction of a robust global sampled-data exponential observer, under additional conditions. The sampled-data exponential observer is robust with respect to perturbations of the sampling schedule and with respect to measurement errors (see also [13–15] for sampled-data observers).

Section 3 studies the second class of systems, with the property of evolving on an open proper subset of $\Re^n$. Here, the "candidate observer" does not guarantee that the observer trajectories lie within the open set, and this is accomplished by adding an appropriate correction term. The design of the correction term is

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performed after transforming the system through an appropriate smooth injective map that maps the open set onto $\Re^n$, even though exponential convergence is enforced in the original coordinates. The results of Section 3 are important because for many classes of systems the state evolves in an open set (for example, biological systems usually evolve in the open first quadrant). However, there is another reason that motivates the results of Section 3. If a change of coordinates $X = \Phi(x)$ can be found, where $\Phi : \Re^n \to \Re^n$ is a smooth injective mapping satisfying $D\Phi(x)f(x,u) = A\Phi(x) + b(h(x),u)$ for all $x \in \Re^n$ for certain Hurwitz matrix $A \in \Re^{n\times n}$ and certain mapping $b : h(\Re^n) \times U \to \Re^n$, then the mapping $\Phi : \Re^n \to \Re^n$ can be used for the design of an observer for (1.1) and (1.2) under additional hypotheses (see [2,8]). The results of Section 3 show that we do not have to assume that $\Phi(\Re^n) = \Re^n$ (i.e., $\Phi : \Re^n \to \Re^n$ is onto); instead, we can require that $A = \Phi(\Re^n)$ is an open set and apply Theorem 3.1.

Finally, in Section 4, we present two illustrative examples of application of the general results. The first example is a system with monotone nonlinearities, and we apply the results of Section 2 to derive a global exponential observer, first under continuous-time measurements and subsequently under sampled measurements. The second example is a bioreactor, following the chemostat model, with positive state variables evolving on the open first quadrant of $\Re^2$. Applying the results of Section 3 leads to a global exponential observer, with positive state estimates. The Appendix contains the proofs of useful technical results.

Notation. Throughout this paper, we adopt the following notation.

* $\Re_+ := [0, +\infty)$.
* By $C^0(A;\Omega)$, we denote the class of continuous functions on $A \subseteq \Re^n$, which take values in $\Omega \subseteq \Re^m$. By $C^k(A;\Omega)$, where $k \geq 1$ is an integer, we denote the class of functions on $A \subseteq \Re^n$ with continuous derivatives of order k, which take values in $\Omega \subseteq \Re^m$.
* By $\mathrm{int}(A)$, we denote the interior of the set $A \subseteq \Re^n$.
* For a vector $x \in \Re^n$, we denote by x' its transpose and by $|x|$ its Euclidean norm. $A' \in \Re^{n\times m}$ denotes the transpose of the matrix $A \in \Re^{m\times n}$ and $|A|$ denotes the induced norm of the matrix $A \in \Re^{m\times n}$, i.e., $|A| = \sup\{|Ax| : x \in \Re^m, |x| = 1\}$.
* A function $V : \Re^n \to \Re_+$ will be called positive definite if $V(0) = 0$ and $V(x) > 0$ for all $x \neq 0$. A function $V : \Re^n \to \Re_+$ will be called radially unbounded if the sets $\{x \in \Re^n : V(x) \leq M\}$ are either empty or bounded for all $M \geq 0$.
* For a function $V \in C^1(A;\Re)$, the gradient of V at $x \in A \subseteq \Re^n$, denoted by $\nabla V(x)$, is the row vector $\nabla V(x) = \left[\frac{\partial V}{\partial x_1}(x) \quad \cdots \quad \frac{\partial V}{\partial x_n}(x)\right]$.

2. Systems with a globally asymptotically stable compact set

Consider the forward complete system (1.1) and (1.2). Our main hypothesis in this section guarantees that there exists a compact set which is robustly globally asymptotically stable (the adjective *robust* means uniformity to all measurable and locally essentially bounded inputs $u : \Re_+ \to U$).

(H1) *There exist a radially unbounded (but not necessarily positive definite) function $V \in C^2(\Re^n;\Re_+)$, a positive definite function $W \in C^1(\Re^n;\Re_+)$, and a constant $R > 0$ such that the following inequality holds for all $(x,u) \in \Re^n \times U$ with $V(x) \geq R$:*

$$\nabla V(x) f(x,u) \leq -W(x). \tag{2.1}$$

Indeed, hypothesis (H1) guarantees that, for every initial condition $x(0) \in \Re^n$, and for every measurable and locally essentially bounded input $u : \Re_+ \to U$, the solution $x(t)$ of (1.1) enters the compact set $S = \{x \in \Re^n : V(x) \leq R\}$ after a finite transient period, i.e., there exists $T \in C^0(\Re^n;\Re_+)$ such that $x(t) \in S$, for all $t \geq T(x(0))$. Moreover, notice that the compact set $S = \{x \in \Re^n : V(x) \leq R\}$ is positively invariant. This fact is guaranteed by the following lemma, which is proved in the Appendix.

Lemma 2.1. *Consider system (1.1) under hypothesis (H1). Then there exists $T \in C^0(\Re^n;\Re_+)$ such that, for every $x_0 \in \Re^n$, and for every measurable and locally essentially bounded input $u : \Re_+ \to U$, the solution $x(t) \in \Re^n$ of (1.1) with initial condition $x(0) = x_0$ and corresponding to input $u : \Re_+ \to U$ satisfies $V(x(t)) \leq \max(V(x_0), R)$ for all $t \geq 0$ and $V(x(t)) \leq R$ for all $t \geq T(x_0)$.*

Our second hypothesis guarantees that we are in a position to construct an appropriate local exponential observer for system (1.1) and (1.2).

(H2) *There exist a symmetric and positive definite matrix $P \in \Re^{n\times n}$, constants $\mu > 0$, $b > R$, and a smooth mapping $k : \Re^n \times h(\Re^n) \times U \to \Re^n$ with $k(\xi,y,u) = 0$ for all $(\xi,y,u) \in \Re^n \times h(\Re^n) \times U$ with $h(\xi) = y$ such that the following inequality holds:*

$$\begin{aligned}&(\xi - x)'P\left(f(\xi,u) + k(\xi,h(x),u) - f(x,u)\right)\\ &\quad \leq -\mu|\xi - x|^2, \quad \text{for all } u \in U,\ \xi, x \in \Re^n \text{ with } V(\xi) \leq b\\ &\qquad \text{and } V(x) \leq R.\end{aligned} \tag{2.2}$$

Indeed, hypothesis (H2) in conjunction with hypothesis (H1) guarantees that, for every $x(0) \in S = \{x \in \Re^n : V(x) \leq R\}$, and for every measurable and locally essentially bounded input $u : \Re_+ \to U$, the solution of system (1.1) and (1.2) with

$$\dot{\xi} = f(\xi,u) + k(\xi,y,u) \tag{2.3}$$

will satisfy an estimate of the form $|\xi(t) - x(t)| \leq M\exp(-\sigma t)|\xi(0) - x(0)|$, for all $t \geq 0$ for appropriate constants $M, \sigma > 0$, provided that the initial estimation error $|\xi(0) - x(0)|$ is sufficiently small. This is why system (2.3) is termed "a local exponential observer". The reader should notice that hypothesis (H2) holds automatically for nonlinear systems of the form

$$\begin{aligned}\dot{x}_1 &= f_1(x_1) + x_2\\ \dot{x}_2 &= f_2(x_1,x_2) + x_3\\ &\vdots\\ \dot{x}_n &= f_n(x_1,\ldots,x_n) + u\\ y &= x_1\end{aligned} \tag{2.4}$$

for every $b > R > 0$ and for every non-empty set $U \subseteq \Re^m$, where $f_i : \Re^i \to \Re$ $(i = 1,\ldots,n)$ are smooth mappings.

In order to be able to construct a nonlinear exponential observer for system (1.1) and (1.2), we need an additional technical hypothesis.

(H3) *There exist constants $c \in (0,1)$, $R \leq a < b$ such that the following inequality holds:*

$$\begin{aligned}&\nabla V(\xi)\left(f(\xi,u) + k(\xi,h(x),u)\right)\\ &\quad \leq -W(\xi) + (1-c)\left|\nabla V(\xi)\right|^2\\ &\quad \times \frac{(\xi - x)'P\left(f(\xi,u) + k(\xi,h(x),u) - f(x,u)\right)}{\nabla V(\xi)P(\xi - x)}\\ &\qquad \text{for all } u \in U,\ \xi, x \in \Re^n \text{ with } a < V(\xi) \leq b,\\ &\qquad \nabla V(\xi)P(\xi - x) < 0 \text{ and } V(x) \leq R.\end{aligned} \tag{2.5}$$

Hypothesis (H3) imposes constraints for the evolution of the trajectories of the local observer (2.3). Indeed, inequality (2.5) imposes a bound on the derivative of the Lyapunov function $V \in C^1(\Re^n;\Re_+)$ along the trajectories of the local observer (2.3) for specific regions of the state space.

We are now ready to state and prove the main result of the present section.