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# A new method for solving Bezout equations over 2-D polynomial matrices from delay systems

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ABSTRACT

In the algebraic system theory of delay systems, it is well known that under spectral controllability or canonicity, a Bezout equation set up with a coprime pair of 2-D polynomial matrices has a solution in polynomial matrices with coefficient belonging to a ring of entire functions. We propose a new method for solving such Bezout equations. The basic concept involves the reduction of a Bezout equation over 2-D polynomial matrices to a simple scalar equation over 1-D polynomials. Due to the basic concept, it can be used to calculate a solution even by hand and is particularly efficient in the absence of modern computer algebra systems.



## 1. Introduction

One important application of multidimensional system theory over two-dimensional (2-D) polynomial matrices is delay system theory, in which the two variables are the Laplace operator $s$ and the delay operator $z$ with the delay-time $L$. Unlike the general 2-D system theory, delay system theory uses the relation $z(s) = e^{-Ls}$ between the two variables. Based on this relation, a ring of entire functions was introduced in [1] for deriving an advanced control of delay systems, such as finite spectrum assignment [1,2]. It was shown in [2] that under spectral controllability or canonicity [3], a Bezout equation set up with a coprime pair of 2-D polynomial matrices has a solution in polynomial matrices with coefficients belonging to a ring of entire functions. In contrast, in the general 2-D case, the solvability of the Bezout equation over the ring of polynomial matrices requires zero coprimeness [4], which is stronger than minor and factor coprimeness [4–6].

A solution to the Bezout equation using the ring of entire functions has been widely used in the control design for delay systems. For example, its application to state feedback design was presented in [2]. In addition, a solution to the Bezout equation has been used not only in finite spectrum assignment, such as in [7,8], but also in repetitive control [9].

This paper proposes a new method for solving such Bezout equations over 2-D polynomial matrices, in which their solutions are obtained in 2-D polynomial matrices with coefficients belonging to the ring of entire functions. The basic concept of the proposed method is that we reduce a Bezout equation over 2-D polynomial matrices to a simple scalar equation over one-dimensional (1-D) polynomials.

As the dimensions of a Bezout equation (i.e., the dimensions of an identity matrix on the right-hand side) increase, the difficulty and amount of matrix calculations increases in general. However, due to the basic concept, the proposed method does not directly depend on the dimensions of the Bezout equation. That is, we reduce calculations on 2-D polynomial matrices to simple ones on 1-D polynomials. Thus, we can simplify calculations themselves and can reduce the amount of the calculations.

Generally speaking, a Bezout equation over 2-D polynomial matrices becomes solvable over polynomials by the introduction of auxiliary variables, such as the entire functions in this paper, which allow us to divide by a "key polynomial" in $s$ using Gröbner bases [10]. As the degree of the key polynomial increases, the number of the auxiliary variables increases. Thus, the difficulty of matrix calculations increases, and modern computer algebra systems such as SINGULAR [10,11] are of absolute necessity. However, the proposed method does not directly depend on the degree of the key polynomial. It can be used to calculate a solution even by hand and is particularly efficient in the absence of such modern computer algebra systems.

During the last 20 years, no new solution methods for Bezout equations over 2-D polynomial matrices have been published in the field of control theory. Significant effort has been spent on applications of solutions to such Bezout equations to control system design rather than on improvements of the existing

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solution methods. On the other hand, especially in the field of polynomial algebra, significant effort has been spent on finding solution methods for a more general class of algebraic equations. The Bezout equations considered in this paper can be solved by such methods, e.g., Gröbner basis methods. Moreover, modern computer algebra systems that assist polynomial computations have also been developed, e.g., SINGULAR.

Against such a background, the main contribution of this paper is the expansion of the selection scope of solution methods. Conventional methods (including Gröbner basis methods) are used for directly solving a given Bezout equation. On the other hand, the proposed method is based on the reduction of a given Bezout equation over 2-D polynomial matrices to a simple scalar equation over 1-D polynomials. This is a fundamentally new approach, which can supplement the solution methods for a general class of algebraic equations in the field of polynomial algebra. For example, Gröbner basis methods are now only used for obtaining the indispensable equation for the reduction (see Section 4.1).

The following notations are used in this paper: $\mathbb{R}$ and $\mathbb{C}$: the fields of real and complex numbers. $i$: the imaginary unit. $\bar{c}$: the complex conjugate of $c \in \mathbb{C}$. $\mathcal{Y}[x_1, x_2]$: the ring of polynomials in $x_1$ and $x_2$ with coefficients in $\mathcal{Y}$. $\mathcal{Y}^{\ell\times m}$: the set of $\ell \times m$ matrices with elements in $\mathcal{Y}$. $\mathbb{R}(x)$: the field of real rational functions in $x$. $\mathbb{R}(x)[y]$: the ring of polynomials in $y$ with coefficients in $\mathbb{R}(x)$. $\deg_x$: the degree in $x$ of a polynomial. $\boldsymbol{E}$: an identity matrix of appropriate dimensions. $\boldsymbol{0}$: a zero matrix of appropriate dimensions. ${}^t\boldsymbol{A}$: the transposed matrix of $\boldsymbol{A}$.

## 2. Problem statement

### 2.1. Minor coprimeness of 2-D polynomial matrices

We first define the minor left coprimeness of 2-D polynomial matrices, which is the basis of the discussion.

**Definition 2.1** ([4]). Suppose $\boldsymbol{P} \in (\mathbb{R}[s,z])^{\ell\times\ell}$ and $\boldsymbol{Q} \in (\mathbb{R}[s,z])^{\ell\times m}$. $(\boldsymbol{P},\boldsymbol{Q})$ is said to be a "minor left coprime pair" if the greatest common polynomial divisor (gcd) of all the $\ell\times\ell$ minors of $\begin{bmatrix}\boldsymbol{P} & \boldsymbol{Q}\end{bmatrix}$ is a nonzero constant. Suppose $\boldsymbol{M} \in (\mathbb{R}[s,z])^{\ell\times m}$ and $\boldsymbol{N} \in (\mathbb{R}[s,z])^{m\times m}$. $(\boldsymbol{M},\boldsymbol{N})$ is said to be a "minor right coprime pair" if $({}^t\boldsymbol{M}, {}^t\boldsymbol{N})$ is a minor left coprime pair.

The variables $s$ and $z$ are assumed to be the Laplace operator and the delay operator with the delay-time $L$, i.e., $z : f(t) \longrightarrow f(t-L)$, respectively.

The following proposition demonstrates how minor coprimeness of $(\boldsymbol{P},\boldsymbol{Q})$ is related to elementary operations on the matrix $\begin{bmatrix}\boldsymbol{P} & \boldsymbol{Q}\end{bmatrix}$:

**Proposition 2.1** ([4]). *Suppose* $\boldsymbol{P} \in (\mathbb{R}[s,z])^{\ell\times\ell}$ *and* $\boldsymbol{Q} \in (\mathbb{R}[s,z])^{\ell\times m}$. $(\boldsymbol{P},\boldsymbol{Q})$ *is a minor left coprime pair if and only if the following conditions are simultaneously satisfied:*

(i) *There exist* $\boldsymbol{U_1} \in (\mathbb{R}[s,z])^{\ell\times\ell}$, $\boldsymbol{U_2} \in (\mathbb{R}[s,z])^{m\times\ell}$, *and* $\alpha\ (\neq 0) \in \mathbb{R}[s]$ *such that*

$$\boldsymbol{PU_1} + \boldsymbol{QU_2} = \alpha\boldsymbol{E}. \tag{2.1}$$

(ii) *There exist* $\boldsymbol{V_1} \in (\mathbb{R}[s,z])^{\ell\times\ell}$, $\boldsymbol{V_2} \in (\mathbb{R}[s,z])^{m\times\ell}$, *and* $\beta\ (\neq 0) \in \mathbb{R}[z]$ *such that*

$$\boldsymbol{PV_1} + \boldsymbol{QV_2} = \beta\boldsymbol{E}. \tag{2.2}$$

We assume that $\alpha$ and $\beta$ are the monic polynomials of minimal degree satisfying (2.1) and (2.2), respectively. The matrix $\begin{bmatrix}\boldsymbol{P} & \boldsymbol{Q}\end{bmatrix}$ can be reduced to $\begin{bmatrix}\boldsymbol{E} & \boldsymbol{0}\end{bmatrix}$ through finitely many elementary operations over $\mathbb{R}(s)[z]$ and $\mathbb{R}(z)[s]$ to obtain (2.1) and (2.2), respectively.

Generally speaking, a Bezout equation over 2-D polynomial matrices becomes solvable over polynomials by the introduction of auxiliary variables, such as the entire functions in Section 2.2, which allow us to divide by a key polynomial in $s$ in a specific way. The $\alpha$ in (2.1) is the key polynomial in this case.

### 2.2. Ring of entire functions

Define

$$\Theta_0 = \left\{ a,\ \frac{d^k\theta_a}{ds^k},\ \frac{d^k}{ds^k}\left(\theta_c + \theta_{\bar{c}}\right),\ \frac{d^k}{ds^k}\left\{i(\theta_c - \theta_{\bar{c}})\right\} \,\middle|\, a \in \mathbb{R},\ c \in \mathbb{C}\setminus\mathbb{R},\ k = 0, 1, 2, \ldots \right\}, \tag{2.3}$$

where $\theta_c$ is the entire function

$$\theta_c = \frac{1 - e^{-L(s-c)}}{s-c} = \frac{1 - e^{Lc}z(s)}{s-c}, \quad c \in \mathbb{C} \tag{2.4}$$

and $z(s) = e^{-Ls}$ is the Laplace transform of the delay operator $z$. Let $\Theta$ denote the commutative ring generated by $\Theta_0$ with obvious addition and multiplication. The ring $\Theta$ was used in [2].

### 2.3. Bezout equation

We define the problem as follows.

**Problem 2.1.** Suppose that $\boldsymbol{P} \in (\mathbb{R}[s,z])^{\ell\times\ell}$, $\boldsymbol{Q} \in (\mathbb{R}[s,z])^{\ell\times m}$, and $(\boldsymbol{P},\boldsymbol{Q})$ is a minor left coprime pair. In addition, suppose that $(\boldsymbol{P},\boldsymbol{Q})$ satisfies

$$\operatorname{rank}\begin{bmatrix}\boldsymbol{P}(s, e^{-Ls}) & \boldsymbol{Q}(s, e^{-Ls})\end{bmatrix} = \ell, \quad \forall s \in \mathbb{C}. \tag{2.5}$$

Then, find a solution $(\boldsymbol{X},\boldsymbol{Y})$ to the Bezout equation

$$\boldsymbol{P}(s, e^{-Ls})\boldsymbol{X}(s, e^{-Ls}) + \boldsymbol{Q}(s, e^{-Ls})\boldsymbol{Y}(s, e^{-Ls}) = \boldsymbol{E}, \tag{2.6}$$

where $\boldsymbol{X} \in (\Theta[s,z])^{\ell\times\ell}$, and $\boldsymbol{Y} \in (\Theta[s,z])^{m\times\ell}$.

Consider a delay system with the state–space equation

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \tag{2.7}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^\ell$ is the state variable vector, $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the input, $\boldsymbol{A} \in (\mathbb{R}[z])^{\ell\times\ell}$, and $\boldsymbol{B} \in (\mathbb{R}[z])^{\ell\times m}$. Condition (2.5) with $(\boldsymbol{P},\boldsymbol{Q}) = (s\boldsymbol{E} - \boldsymbol{A}, \boldsymbol{B})$ implies that the system (2.7) is spectrally controllable [2].

Also, consider a left coprime factorization $\boldsymbol{D}^{-1}\boldsymbol{N}$ of the input–output transfer function of a delay system, where $\boldsymbol{D} \in (\mathbb{R}[z])^{\ell\times\ell}$ and $\boldsymbol{N} \in (\mathbb{R}[z])^{\ell\times m}$. This factorization corresponds to the input–output relation

$$\boldsymbol{D}\boldsymbol{y}(t) = \boldsymbol{N}\boldsymbol{u}(t), \tag{2.8}$$

where $\boldsymbol{y}(t) \in \mathbb{R}^\ell$ is the output and $\boldsymbol{u}(t) \in \mathbb{R}^m$ is the input. Condition (2.5) with $(\boldsymbol{P},\boldsymbol{Q}) = (\boldsymbol{D},\boldsymbol{N})$ implies that the system with the input–output relation (2.8) is spectrally canonical [3].

### 2.4. Related zero pairs and rank condition

We first introduce the following definition:

**Definition 2.2.** For a given minor left coprime pair $(\boldsymbol{P},\boldsymbol{Q})$, consider (2.1) and (2.2). $(s, z) = (s_j, e^{-Ls_j})$ is called a "related zero pair" if it satisfies $\alpha(s_j) = 0$ and $\beta(e^{-Ls_j}) = 0$, where $s_j \in \mathbb{C}$ and $j = 1, 2, \ldots$.

The following proposition implies that Condition (2.5) in the entire complex plane can be reduced to the rank condition on finitely many possible related zero pairs. This proposition is indispensable for clarifying the basic concept of the proposed method.