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Time optimal control of semilinear parabolic equations via bilinear controls☆

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ABSTRACT

We establish the existence of the time optimal control for semilinear parabolic equations with gradient quadratic growth via bilinear controls. It is worth pointing out that there is no restriction on the growth of the nonlinearity $f(s)$ with respect to the variable s in the equation, which is a remarkable difference compared to the semilinear parabolic system with locally distributed controls. The technique used in this paper is the combination of the Hopf–Cole transformation, the a prior estimates on solutions of parabolic equations and the strategy of the stepwise control.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ with C^2 boundary $\partial\Omega$ and $\omega \subset \Omega$ be a nonempty subdomain. Consider the following controlled semilinear parabolic system:

$$\begin{cases} y_t - \Delta y + a|\nabla y|^2 = 1_\omega u(f(y) - \theta) & \text{in } Q_\infty, \\ y(x,t) = 0 & \text{on } \Sigma_\infty, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $Q_\infty = \Omega \times (0, \infty)$, $\Sigma_\infty = \partial\Omega \times (0, \infty)$, a is a constant, 1_ω is the characteristic function of ω, $y_0 \in L^\infty(\Omega)$, $\theta \in L^\infty(Q_\infty)$ and $f \in C(\mathbb{R})$ are given functions.

Eq. (1.1) with $u \equiv 0$ arises in stochastic optimal control theory (see [1, p. 194]). In the context of heat-transfer the term $u(x,t)(y(x,t) - \theta(x,t))$ is used to describe the heat exchange at point (x,t) of the given substance with the surrounding medium of temperature θ according to Newton's Law (see [2, pp. 155–156]).

Let u be a control taken from a given set

$$U_\rho = \{u;\ u \in L^\infty(Q_\infty), |u| \le \rho \text{ a.e. in } Q_\infty\}, \tag{1.2}$$

where $\rho > 0$ is an arbitrary but fixed positive constant.

In this paper, we shall study the following time optimal control problem:

(P) $\min\{T;\ y(\cdot, T) = 0$ a.e. in Ω, $u \in U_\rho$ and y is the solution to (1.1) corresponding to $u\}$.

A function $u \in U_\rho$ is called admissible if the corresponding solution y to (1.1) satisfying $y(\cdot, T) = 0$ a.e. in Ω for some $T > 0$. $T^*(\rho) \triangleq \min\{T; y(\cdot,T) = 0 \text{ a.e. in } \Omega, u \in U_\rho\}$ is called the minimal time for (P) and a control $u^* \in U_\rho$ such that $y^*(\cdot, T^*(\rho)) = 0$ a.e. in Ω is called a time optimal control.

The time optimal control problem was studied first for the finite-dimensional case (cf. [3]). Thereafter, the problem was developed to infinite-dimensional controlled systems (cf. [4,5]). In [6], the time optimal control problem for some controlled parabolic variational inequalities was investigated. However, the method used in [6] is suitable only for the case where the control is distributed in the whole domain Ω. In [7], the time optimal control was obtained for the equation

$$y_t - \Delta y + f(y) = 1_\omega u \quad \text{in } Q_\infty, \tag{1.3}$$

where the control u acts only on a local domain ω, and the aim function is the steady-state solution y_e to (1.3), i.e., $-\Delta y_e(x) + f(y_e(x)) = 0$ in Ω and $y_e(x) = 0$ on $\partial\Omega$.

As is shown in [7], the key to get the existence of a time optimal control is to show the existence of an admissible control which is related to a type of controllability of the equation with some kind of control constraint. It is well known that a rather general class of semilinear parabolic systems is approximate and null controllable by locally distributed controls (cf. [8–13] and the references therein).

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Denote $Q_T = \Omega \times (0, T)$ and $\Sigma_T = \partial\Omega \times (0, T)$. In [14], the authors considered the following parabolic system with nonlinear term involving the state and the gradient

$$\begin{cases} y_t - \Delta y + f(y, \nabla y) = 1_\omega u & \text{in } Q_T, \\ y(x,t) = 0 & \text{on } \Sigma_T, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{1.4}$$

where $f(s, p)$ is locally Lipschitz-continuous and can be rewritten as $f(s, p) = f(0, 0) + g(s, p)s + G(s, p) \cdot p$ for some L^∞_{loc} functions g and G. They proved the exact null controllability and the approximate controllability for the system (1.4) under the following hypothesis

$$\lim_{|(s,p)|\to\infty} \frac{|g(s,p)|}{\log^{3/2}(1+|s|+|p|)} = 0$$

and

$$\lim_{|(s,p)|\to\infty} \frac{|G(s,p)|}{\log^{1/2}(1+|s|+|p|)} = 0.$$

Therefore, in order to obtain the existence of the time optimal control for (1.3) in [7], the growth of the nonlinearity $f(s)$ with respect to the variable s should satisfy some constraint conditions.

Note that (1.1) is a bilinear (or multiplicative) control system, in which there is only one control function u, but it acts both on the right-hand side function θ and on the coefficient of state function. This is quite different from the traditionally additive locally distributed control systems (1.3) and (1.4). On the other hand, it is easy to see that (1.1) cannot be included in (1.4) because of the quadratic growth for the gradient.

For bilinear systems, the dependence of the state function with respect to the control function is highly nonlinear. This leads to many difficulties in the study of bilinear control systems. In fact, little is known about the time optimal control problem of (1.1), by our knowledge. As to the controllability problem, we refer to the early papers [15,16] on the abstract, infinite dimensional system. We also refer to the pioneering works by Lenhart and her co-authors on bilinear optimal controls (cf. [17,18]). Among recent achievements, let us mention the results [19–22] and the book of Khapalov [23] (see also the references therein).

Our goal in this paper is to prove the existence of the time optimal control for (1.1) when the bilinear control acts on any local domain $\omega \subset \Omega$. It is worth pointing out that there is no restriction on the growth of the nonlinearity $f(s)$ with respect to the variable s in our proofs, which is a remarkable difference compared to the semilinear systems (1.3) and (1.4).

The paper is organized as follows. In Section 2, we prove the well-posedness of (1.1). The null controllability of (1.1) is established in Section 3. Finally, we obtain the existence of the time optimal control in Section 4.

2. Existence and uniqueness of solutions

Since no growth restriction is imposed on the nonlinearity $f(s)$, in general, (1.1) has no a globally defined (in time) solution, and the solution may blow up in finite time. Therefore, we need to study the existence time of the local solution. Throughout this paper, we investigate the weak solution.

Given $a \neq 0$. Consider the following problem

$$\begin{cases} z_t - \Delta z + a1_\omega uf\left(-\frac{1}{a}\ln z\right)z - a1_\omega u\theta z = 0 & \text{in } Q_T, \\ z(x,t) = 1 & \text{on } \Sigma_T, \\ z(x,0) = e^{-ay_0(x)} & \text{in } \Omega. \end{cases} \tag{2.1}$$

As in [24], we denote by $\mathring{W}_2^{1,0}(Q_T)$ and $\mathring{W}_2^{1,1}(Q_T)$ two standard Sobolev spaces. We say z is a weak solution of (2.1), if $z - 1 \in \mathring{W}_2^{1,0}(Q_T) \cap L^\infty(Q_T)$ and the integral equality

$$\iint_{Q_T} \left[-z\varphi_t + \nabla z \cdot \nabla\varphi + a1_\omega uf\left(-\frac{1}{a}\ln z\right)z\varphi - a1_\omega u\theta z\varphi\right]dxdt = \int_\Omega z(x,0)\varphi(x,0)dx \tag{2.2}$$

is fulfilled for any test function $\varphi \in \mathring{W}_2^{1,0}(Q_T)$ with $\frac{\partial\varphi}{\partial t} \in L^1(Q_T)$ and $\varphi(\cdot, T)|_\Omega = 0$.

Lemma 2.1. *Let $y_0 \in L^\infty(\Omega)$ and $\theta \in L^\infty(Q_\infty)$. Assume that $f \in C(\mathbb{R})$, $f(0) = 0$ and $f(s)$ is Lipschitz continuous on $[-M, M]$ for any $0 < M < \infty$. Then there exists a constant ε_0 depending on f, θ and y_0, such that if $\|u\|_{L^\infty(Q_\infty)} \leq \varepsilon_0$, then (2.1) admits uniquely a weak solution in $Q_{T_0} = \Omega \times (0, T_0)$, which satisfies*

$$0 < M_0 = \operatorname*{ess\,inf}_{Q_{T_0}} z < \operatorname*{ess\,sup}_{Q_{T_0}} z = M_1 < \infty \quad \text{in } Q_{T_0}, \tag{2.3}$$

where T_0 is a constant and $T_0 \geq 1$.

Proof. Consider the following problem

$$\begin{cases} w_t - \Delta w = a1_\omega uF(w) & \text{in } Q_T, \\ w(x,t) = 0 & \text{on } \Sigma_T, \\ w(x,0) = w_0(x) = e^{-ay_0(x)} - 1 & \text{in } \Omega, \end{cases} \tag{2.4}$$

where

$$F(w) = -f\left(-\frac{1}{a}\ln(w+1)\right)(w+1) + \theta(w+1).$$

Clearly, if $w \in \mathring{W}_2^{1,0}(Q_T) \cap L^\infty(Q_T)$ is the weak solution of (2.4), then $z = w + 1$ is the weak solution of (2.1).

Let $\{j_n\}_{n=1}^{+\infty}$ be a standard mollifying sequence in $\mathbb{R}$, namely, $j_n(s) = \frac{1}{n}j(\frac{s}{n})$ with $j(s) \geq 0$, $j(s) \in C_0^\infty(\mathbb{R})$, $\operatorname{supp} j(s) \subset [-1, 1]$ and $\int_{-\infty}^{+\infty} j(s)ds = 1$. For fixed $u, \theta \in L^\infty(Q_\infty)$, consider the following problem

$$\begin{cases} w_t - \Delta w = au_nF_n(w) & \text{in } Q_T, \\ w(x,t) = 0 & \text{on } \Sigma_T, \\ w(x,0) = w_0^{(n)}(x) & \text{in } \Omega. \end{cases} \tag{2.5}$$

We define F_n, u_n, θ_n and $w_0^{(n)}$ as follows. Put

$$F_n(s) = j_n * \max(-n, \min(F(\tau), n)) \triangleq \int_{\mathbb{R}} j_n(s-\tau)\max(-n, \min(F(\tau), n))d\tau.$$

It is easily verified that $F_n \in C^\infty(\mathbb{R})$ and $|F_n| \leq n$. Similarly, we can define u_n and θ_n by using the above convolution operator $*$. For example, define

$$u_n(x,t) = \iint_{\mathbb{R}^{N+1}} \prod_{1\leq i\leq N} j_n(x_i - y_i)j_n(t-\tau) \cdot \max(-n, \min(1_\omega u(y,\tau), n))dyd\tau.$$

Then $u_n \in C^\infty(Q_\infty)$ with $u_n \equiv 0$ in $\omega_n \times \mathbb{R}^+$, where $\omega_n = \{x \in \Omega \setminus \omega;\ \operatorname{dist}(x, \omega) < \frac{1}{n}\}$, and for any $p \in [1, +\infty)$, $\|u_n - 1_\omega u\|_{L^p(Q_\infty)} \to 0$ as $n \to \infty$. Likewise, $\theta_n \in C^\infty(Q_\infty)$, and $\|\theta_n - \theta\|_{L^p(Q_\infty)} \to 0$ as $n \to \infty$.

Let $\hat{w}_{0n}(x) = 0$ if $x \in \Omega$ and $\operatorname{dist}(x, \partial\Omega) < 2/n$, and $\hat{w}_{0n}(x) = w_0(x)$ if $x \in \Omega$ and $\operatorname{dist}(x, \partial\Omega) \geq 2/n$. Denote

$$w_0^{(n)}(x) = \iint_{\mathbb{R}^N} \prod_{1\leq i\leq N} j_n(x_i - y_i)\hat{w}_{0n}(y)dy.$$