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An extension of the invariance principle for dwell-time switched nonlinear systems[☆]

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ABSTRACT

In this paper, we propose an extension of the invariance principle for nonlinear switched systems under dwell-time switched solutions. This extension allows the derivative of an auxiliary function V, also called a Lyapunov-like function, along the solutions of the switched system to be positive on some sets. The results of this paper are useful to estimate attractors of nonlinear switched systems and corresponding basins of attraction. Uniform estimates of attractors and basin of attractions with respect to time-invariant uncertain parameters are also obtained. Results for a common Lyapunov-like function and multiple Lyapunov-like functions are given. Illustrative examples show the potential of the theoretical results in providing information on the asymptotic behavior of nonlinear dynamical switched systems.

1. Introduction

Nonlinear switched systems arise in practice when modeling the operation of many engineering systems [1–4]. Although switching is not a new concept in engineering, in the past decade the theory of switched systems has attracted the attention of many researchers. As a consequence, the stability theory for nonlinear switched systems has significantly developed in this period.

When the subsystems under switching share a common equilibrium point, the existence of a common Lyapunov function, for example, is a sufficient condition for asymptotic stability of the equilibrium point of nonlinear switched systems under arbitrary switching [5]. However, a common Lyapunov function may be difficult to find or may not exist. To overcome this difficulty and avoid conservative results, a multiple Lyapunov function approach has also been considered [6,7].

Despite the important advances in stability theory, the attractor of many switched systems may not be an equilibrium point. A classical example is the on–off temperature control system. For this class of problems, we are not interested in studying the stability of a particular equilibrium point but the asymptotic behavior of solutions.

The invariance principle is a powerful tool to analyze the asymptotic behavior of dynamical system solutions. It was first developed for ordinary differential equations [8,9] and afterward developed for other classes of systems, including discrete systems [10], functional differential equations [11] and switched systems [12–16].

The invariance principle relies on the existence of a Lyapunov-like function satisfying some properties to analyze the asymptotic behavior of system solutions. A key property is the nonpositiveness of the derivative of the Lyapunov-like function along the solutions. The main disadvantage of the invariance principle is that it does not yield any clue on how to find the Lyapunov-like function. Finding such function, satisfying all the assumptions of the invariance principle, may be difficult for many nonlinear dynamical systems. In order to overcome this problem, various extensions of LaSalle's invariance principle have been proposed. An extension of the invariance principle, which allows the derivative of a Lyapunov-like function to be positive on some bounded sets, was proposed for continuous systems in [17], for discrete systems in [10] and for delayed systems in [18]. These results were successfully applied to the problem of synchronization [17] and to obtain estimates of attractors of uncertain dynamical systems [19].

We propose an extension of the results of [17] to dwell-time switched systems composed of a finite number of continuous nonlinear vector fields. The results of this paper are also an extension of the invariance principle for switched systems given in [13]. The advantage of this extension is that Lyapunov-like

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functions are not required to be positive definite and their derivative along the solutions of the switched system can be positive on some sets. Also, the conditions on the Lyapunov-like function along the solutions are less restrictive than those in [13].

2. Preliminaries

In this paper, we study the asymptotic behavior of solutions of the following class of continuous-time switched systems

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \qquad x(0) = x_0 \tag{1}$$

where $\sigma(t) : [0, \infty) \to \mathcal{P} = \{1, 2, \ldots, N\}$ is a piecewise constant function, continuous from the right, called a switching signal and f_p is a $\mathcal{C}^1$ vector field of $\mathbb{R}^n$ for every $p \in \mathcal{P}$. Let $\{\tau_k\}$ be a sequence of consecutive switching times associated to σ and $I_p = \{t \in [\tau_k, \tau_{k+1}) : \sigma(\tau_k) = p\}$ be the union of intervals where system p is active. A continuous piecewise-smooth function $x_{\sigma(t)}(t) : \mathcal{I} \to \mathbb{R}^n$ is a solution of the switched system (1) in the interval $\mathcal{I}$ if $x_{\sigma(t)}(t)$ satisfies $\dot{x}_{\sigma(t)}(t) = f_p(x_{\sigma(t)}(t))$ for every $t \in I_p \cap \mathcal{I}$ for all $p \in \mathcal{P}$. The main results presented in this paper are obtained for bounded solutions, which are defined for all $t \geq 0$. In this case, we assume that the switching sequence τ_k is divergent and that each system p is active infinite times. In other words, we assume for every $T > 0$ and $p \in \mathcal{P}$ the existence of a k such that $\sigma(\tau_k) = p$ and $\tau_k > T$. The set of all switched solutions are denoted by $\mathcal{S}$. We denote $\varphi_{\sigma(t)}(t, x_0)$, or simply $\varphi(t, x_0)$, the solution of (1) starting at x_o at time $t = 0$ under the switching signal $\sigma(t)$.

We study the solutions of system (1) under a particular class of switching signals, that is, the solutions that have a dwell time. For easy reference, some preliminary definitions, which were taken from [13] (see also [20,5]), are presented.

Definition 1 (*Dwell Time*)**.** The solution $\varphi_{\sigma(t)}(t, x_0) \in \mathcal{S}$ of (1) has a nonvanishing dwell time if there exists $h > 0$ such that

$$\inf_k (\tau_{k+1} - \tau_k) \geq h \tag{2}$$

where $\{\tau_k\}$ is the sequence of switching times associated with $\varphi_{\sigma(t)}(t, x_0)$. The number h is called a dwell time for $\varphi(t, x_0)$ and the set of all switched solutions possessing a nonvanishing dwell time is denoted by $\mathcal{S}_{dwell} \subset \mathcal{S}$.

Definition 2 (*Weak Invariance*)**.** A compact set $\mathcal{M}$ is weakly invariant with respect to the switched system (1) if for each $x_0 \in \mathcal{M}$ there exists an index $p \in \mathcal{P}$, a solution $\varphi(t, x_0)$ of the vector field $f_p(x)$ and a real number $b > 0$ such that $\varphi(t, x_0) \in \mathcal{M}$ for either $t \in [-b, 0]$ or $t \in [0, b]$.

A switched solution $\varphi(t, x_0)$ of (1) is attracted to a compact set $\mathcal{M}$ if for each $\epsilon > 0$ there exists a time $T > 0$ such that

$$\varphi(t, x_0) \in B(\mathcal{M}, \epsilon) \quad \text{for } t \geq T \tag{3}$$

where $B(\mathcal{M}, \epsilon) = \cup_{a \in \mathcal{M}} B(a, \epsilon)$. Clearly $\varphi(t, x_0)$ is attracted to $\mathcal{M}$ if and only if

$$\lim_{t \to \infty} dist(\varphi(t, x_0), \mathcal{M}) = 0. \tag{4}$$

Definition 3 (*Limit Points*)**.** Let $\varphi(t, x_0) : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous curve. A point q is a limit point of $\varphi(t, x_0)$ if there exists a sequence $\{t_k\}_{k \in N}$,with $t_k \to \infty$, as $k \to \infty$ such that $\lim_{k\to\infty} \varphi(t_k, x_0) = q$. The set of all limit points of $\varphi(t, x_0)$ will be denoted by $\omega^+(x_0)$.

Proposition 1 (*Properties of Limit Set*)**.** *Let $\varphi(t, x_0) \in \mathcal{S}_{dwell}$ be a bounded switched solution of (1) for $t \geq 0$. Then, $\omega^+(x_0)$ is nonempty, compact and weakly invariant. Moreover, $\varphi(t, x_0)$ is attracted to $\omega^+(x_0)$.*

Proof. See [13]. □

3. Extension of the invariance principle

In this section, extensions of the invariance principle for switched systems are developed. The main feature of these extensions is that the derivative of the Lyapunov-like function V can be positive on some sets. By relaxing the signs of the derivative of the Lyapunov-like function V, it becomes easier to find V satisfying the assumptions of the invariance principle. As a consequence, the asymptotic behavior of a larger class of nonlinear switched dynamical systems can be studied with this theory.

Two extensions of the invariance principle are developed in this paper. One considers a common Lyapunov-like function for all subsystems while the other considers multiple Lyapunov-like functions.

3.1. Common Lyapunov-like function

We first consider the existence of a single Lyapunov-like function V for all subsystems of the switched system (1). Let $C_p = \{x \in \mathbb{R}^n : \nabla V(x) f_p(x) > 0\}$ be the set where the derivative of function V along trajectories of system p is positive and $E_p = \{x \in R^n : \nabla V(x) f_p(x) = 0\}$ the set where this derivative is equal to zero. Define $C = \cup_{p \in \mathcal{P}} C_p$ and $E = \cup_{p \in \mathcal{P}} E_p$ and consider the following lemma that establishes a result on the positive invariance of sublevel sets of the Lyapunov-like function V.

Lemma 1 (*Positive Invariant Set*)**.** *Consider the switched system (1) and let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Let ℓ be a real number satisfying $\sup_{x \in C} V(x) \leq \ell < \infty$. If $x_0 \in \Omega_\ell = \{x \in \mathbb{R}^n : V(x) \leq \ell\}$, then every bounded solution $\varphi(t, x_0) \in \mathcal{S}_{dwell}$ stays inside Ω_ℓ for all $t \geq 0$.*

Proof. Let $x_0 \in \Omega_\ell$ and $\varphi(t, x_o) \in \mathcal{S}_{dwell}$ a bounded solution. Suppose there exists $\tau > 0$ such that $\varphi(\tau, x_0) \notin \Omega_\ell$. Then, there exists $\bar{\tau} \in (0, \tau)$ such that $V(\varphi(\bar{\tau}, x_0)) = \ell$ (by the continuity of V and $\varphi(t, x_o)$) and $V(\varphi(t, x_0)) > \ell, \forall t \in (\bar{\tau}, \tau]$, but this is a contradiction since $\varphi(t, x_0) \in \mathcal{S}_{dwell}$ and $\nabla V(x) f_p(x(t)) \leq 0$ out of $\Omega_\ell, \forall p \in \mathcal{P}$. Therefore, $\varphi(t, x_0) \in \Omega_\ell, \forall t \geq 0$. □

Exploiting Lemma 1, we establish the following extension of the invariance principle for switched systems.

Theorem 1 (*Extension of the Invariance Principle*)**.** *Consider the switched system (1) and let $V(x) : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$ function. Suppose the existence of a real number ℓ satisfying $\sup_{x \in C} V(x) \leq \ell < \infty$ and let $\Omega_\ell = \{x \in \mathbb{R}^n : V(x) \leq \ell\}$. Finally, let $\mathcal{M}$ be the union of all the weakly invariant sets which are contained in $E \cup \Omega_\ell$. Then, every bounded solution $\varphi(t, x_0) \in \mathcal{S}_{dwell}$ is attracted to $\mathcal{M}$.*

Proof. First, let $x_0 \in \Omega_\ell$ and $\varphi(t, x_0) \in \mathcal{S}_{dwell}$ be a bounded solution. By Lemma 1 we can prove that $\varphi(t, x_0) \in \Omega_\ell$ for $t \geq 0$. By Proposition 1, $\omega^+(x_0)$ is nonempty, compact, weakly invariant and $\omega^+(x_0) \subset \Omega_\ell$. Moreover, $\varphi(t, x_0)$ is attracted to $\omega^+(x_0)$. Then the solution is attracted to a weakly invariant set inside Ω_ℓ.

Now, let $x_0 \notin \Omega_\ell$ and $\varphi(t, x_0) \in \mathcal{S}_{dwell}$ be a bounded solution. If $\varphi(t, x_0)$ enters Ω_ℓ at some t, then the result follows from the first part of this proof. Suppose the bounded solution $\varphi(t, x_0) \notin \Omega_\ell, \forall t \geq 0$. Since $\ell \geq \sup_{x \in C} V(x)$, $\varphi(t, x_0) \notin C \subset \Omega_\ell, \forall t \geq 0$, that is, $V(\varphi(t, x_0)) > \ell$ and $\nabla V(\varphi(t, x_0)) f_p(\varphi(t, x_0)) \leq 0, \forall t \geq 0, \forall p \in \mathcal{P}$. We conclude that $V(\varphi(t, x_0))$ is a lower bounded non-increasing function of t. Then, there exists $r \in \mathbb{R}$ such that $r = \lim_{t\to\infty} V(\varphi(t, x_0))$. Since the solution is bounded, $\omega^+(x_0)$ is nonempty. Let $a \in \omega^+(x_0)$, then there exists a sequence $\{t_k\}$ with $t_k \to \infty$ as $k \to \infty$ such that $\varphi(t_k, x_0) \to a$. The continuity of V ensures that $V(\varphi(t_k, x_0)) \to V(a)$ as $k \to \infty$, then $V(a) = r$, $\forall a \in \omega^+(x_0)$.