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A comparison of hybridized and standard DG methods for target-based *hp*-adaptive simulation of compressible flow

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ABSTRACT

Objective: We present a comparison between hybridized and non-hybridized discontinuous Galerkin methods in the context of target-based *hp*-adaptation for compressible flow problems. The aim is to provide a critical assessment of the computational efficiency of hybridized DG methods.

Method: Hybridization of finite element discretizations has the main advantage, that the resulting set of algebraic equations has globally coupled degrees of freedom only on the skeleton of the computational mesh. Consequently, solving for these degrees of freedom involves the solution of a potentially much smaller system. This not only reduces storage requirements, but also allows for a faster solution with iterative solvers. Using a discrete-adjoint approach, sensitivities with respect to output functionals are computed to drive the adaptation. From the error distribution given by the adjoint-based error estimator, *h*- or *p*-refinement is chosen based on the smoothness of the solution which can be quantified by properly-chosen smoothness indicators.

Results: Numerical results are shown for subsonic, transonic, and supersonic flow around the NACA0012 airfoil. *hp*-adaptation proves to be superior to pure *h*-adaptation if discontinuous or singular flow features are involved. In all cases, a higher polynomial degree turns out to be beneficial. We show that for polynomial degree of approximation $p = 2$ and higher, and for a broad range of test cases, HDG performs better than DG in terms of runtime and memory requirements.

1. Introduction

During the last years, discontinuous Galerkin (DG) methods (see, e.g., [1–3]) have become increasingly popular. This is indisputable due to their advantages—high-order accuracy on unstructured meshes, a variational setting, and local conservation, just to name a few.

However, the use of discontinuous function spaces is at the same time the reason for a major disadvantage: unlike in continuous Galerkin (CG) methods, degrees of freedom are not shared between elements. As a consequence, the number of unknowns is substantially higher compared to a CG discretization. Especially for implicit time discretization this imposes large memory requirements, and potentially leads to increased time-to-solution.

In order to avoid these disadvantages, a technique called hybridization may be utilized (see [4–10]), resulting in hybridized discontinuous Galerkin (HDG) methods. Here, the globally coupled unknowns have support on the mesh skeleton, i.e. the element interfaces, only. This reduces the size of the global system and coincidentally improves the sparsity pattern.

However, aiming at industry applications, e.g. turbulent flow around a complete airplane or within an aircraft engine, hybridization alone does most likely not provide a sufficiently successful overall algorithm. In these applications one is usually interested in certain quantities only, for example lift or drag coefficients in aerospace, instead of the solution quality per se. Thus, it might be beneficial to distribute the degrees of freedom within the computational domain in such a way that the solution to the discretized problem is close to optimal with respect to the accuracy of these quantities. To achieve this goal, target-based error control methods have been developed (see [11–15]). One such method is based on the adjoint solution of the original governing equations with respect to the target functional. In this method, an additional linear system of partial differential equations is solved which then gives an estimate on the spatial error distribution contributing to the error in the target functional. This estimate can be used as a criterion for local adaptation. Within the context of low order

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schemes, mesh refinement is used for adaptation [12,13]. Using DG (or HDG), however, offers the additional possibility of varying the polynomial degree within each element. For smooth solutions, this is more efficient compared to mesh refinement, as it yields exponential convergence. In the context of wave problems, Giorgiani et al. [16] showed the benefit of using p-adaptation within an HDG-framework. Combining both mesh- and order-refinement results in so-called hp-adaptation.

In [17], we presented a discretization method for nonlinear convection–diffusion equations. The method is based on a discontinuous Galerkin discretization for convection terms, and a mixed method using H(div) spaces for the diffusive terms. Furthermore, hybridization is used to reduce the number of globally coupled degrees of freedom. Adjoint consistency was shown in [18]. In [19,20], we extended our computational framework to include HDG schemes, as well as adjoint-based h- and hp-adaptation. In the current paper, we compare our HDG method with a standard DG method in the context of hp-adaptation for stationary compressible flow, mainly with the aim to assess the efficiency of both methods.

This paper is structured as follows. We briefly cover the governing equations, namely the compressible Euler and Navier–Stokes equations, in Section 2. After that we introduce our discretization and describe the concept of hybridization in Section 3. In Section 4 we establish the adjoint formulation and show how hybridization can be applied to the dual problem. Then we show its efficiency and robustness with examples from compressible flow, including the subsonic, transonic, and supersonic regime, in Section 5. Finally, we offer conclusions and outlook on future work in Section 6.

2. Governing equations

We consider systems of partial differential equations

$$\nabla \cdot (\mathbf{f}_c(w) - \mathbf{f}_v(w, \nabla w)) = s(w, \nabla w) \tag{1}$$

with convective and diffusive fluxes

$$\mathbf{f}_c : \mathbb{R}^m \to \mathbb{R}^{m\times d} \quad \text{and} \quad \mathbf{f}_v : \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^{m\times d}, \tag{2}$$

respectively, and a state-dependent source term

$$s : \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m \tag{3}$$

on domain $\Omega \subset \mathbb{R}^d$. Potentially, some of these quantities could be zero. We denote the spatial dimension by d and the number of conservative variables by m. Boundary conditions can be applied either to the conservative variables $w \in \mathbb{R}^m$ and their gradient $\nabla w \in \mathbb{R}^{m\times d}$ or directly to the fluxes $\mathbf{f}_c$ and $\mathbf{f}_v$.

2.1. Two-dimensional Euler equations

The Euler equations are comprised of the inviscid compressible continuity, momentum and energy equations. They are given in conservative form as

$$\nabla \cdot \mathbf{f}_c(w) = 0 \tag{4}$$

with the vector of conserved variables

$$w = (\rho, \rho\mathbf{v}, E)^T \tag{5}$$

where ρ is the density, $\mathbf{v}$ is the velocity vector $\mathbf{v} := (v_x, v_y)^T$, and E the total energy. The convective flux is given by

$$\mathbf{f}_c = (\rho\mathbf{v}, p\mathbf{Id} + \mathbf{v}\otimes\mathbf{v}, \mathbf{v}(E+p))^T. \tag{6}$$

Pressure is related to the conservative flow variables w by the equation of state

$$p = (\gamma - 1)\left(E - \frac{1}{2}\rho\mathbf{v}\cdot\mathbf{v}\right) \tag{7}$$

where $\gamma = c_p/c_v$ is the ratio of specific heats, generally taken as 1.4 for air.

Along wall boundaries we apply the slip boundary condition

$$v_n(w) := \mathbf{v}\cdot\mathbf{n} = 0. \tag{8}$$

We also define a boundary function which satisfies $v_n(w_{\partial\Omega}(w)) = 0$ as

$$w_{\partial\Omega}(w) = \begin{pmatrix} 1 & \mathbf{0}^T & 0 \\ \mathbf{0} & \mathbf{Id} - \mathbf{n}\otimes\mathbf{n} & \mathbf{0} \\ 0 & \mathbf{0}^T & 1 \end{pmatrix} w. \tag{9}$$

Prescribing boundary conditions at the far-field can be realized with the aid of characteristic upwinding [21]. Here, the normal convective flux Jacobian is decomposed as

$$\mathbf{f}_c'(w)\cdot\mathbf{n} = Q(w,\mathbf{n})\cdot\Lambda(w,\mathbf{n})\cdot Q^{-1}(w,\mathbf{n}) \tag{10}$$

with $\Lambda(w,\mathbf{n})$ being a diagonal matrix containing the eigenvalues of $\mathbf{f}_c'(w)\cdot\mathbf{n}$. The corresponding right eigenvectors can be found in the columns of $Q(w,\mathbf{n})$. The interior and far-field states in characteristic variables are then given by $w_c = Q(w,\mathbf{n})w$ and $w_{c,\infty} = Q(w,\mathbf{n})w_\infty$, respectively. Finally, depending on the sign of $(\Lambda(w,\mathbf{n}))_{i,i}$, we can construct a boundary state

$$(w_{\partial\Omega}(w))_i = \begin{cases} (Q(w,\mathbf{n})w_c)_i, & (\Lambda(w,\mathbf{n}))_{i,i} \geqslant 0 \\ (Q(w,\mathbf{n})w_{c,\infty})_i, & (\Lambda(w,\mathbf{n}))_{i,i} < 0. \end{cases} \tag{11}$$

2.2. Two-dimensional Navier–Stokes equations

The Navier–Stokes equations in conservative form are given by

$$\nabla\cdot(\mathbf{f}_c(w) - \mathbf{f}_v(w,\nabla w)) = 0. \tag{12}$$

The convective part $\mathbf{f}_c$ of the Navier–Stokes equations coincides with the Euler equations. The viscous flux is given by

$$\mathbf{f}_v = (\mathbf{0}, \boldsymbol{\tau}, \boldsymbol{\tau}\mathbf{v} + k\nabla T)^T. \tag{13}$$

The temperature is defined via the ideal gas law

$$T = \frac{\mu\gamma}{k\cdot\mathrm{Pr}}\left(\frac{E}{\rho} - \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) = \frac{1}{(\gamma-1)c_v}\frac{p}{\rho} \tag{14}$$

where $\mathrm{Pr} = \frac{\mu c_p}{k}$ is the Prandtl number, which for air at moderate conditions can be taken as a constant with a value of $\mathrm{Pr} \approx 0.72$. k denotes the thermal conductivity coefficient. For a Newtonian fluid, the stress tensor is defined as

$$\boldsymbol{\tau} = \mu\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T - \frac{2}{3}(\nabla\cdot\mathbf{v})\mathbf{Id}\right). \tag{15}$$

The variation of the molecular viscosity μ as a function of temperature is determined by Sutherland's law as

$$\mu = \frac{C_1 T^{3/2}}{T + C_2} \tag{16}$$

with $C_1 = 1.458e-6\ \frac{\mathrm{kg}}{\mathrm{ms}\sqrt{\mathrm{K}}}$ and $C_2 = 110.4\ \mathrm{K}$.

Along wall boundaries, we apply the no-slip boundary condition, i.e.

$$\mathbf{v} = \mathbf{0} \tag{17}$$

with corresponding boundary function

$$w_{\partial\Omega}(w) = (\rho, \mathbf{0}, E)^T. \tag{18}$$

Furthermore, one has to give boundary conditions for the temperature. In the present work we use the adiabatic wall condition, i.e.

$$\nabla T\cdot\mathbf{n} = 0. \tag{19}$$