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A DPG method for steady viscous compressible flow

Jesse Chan [a,*], Leszek Demkowicz [b], Robert Moser [b]

[a] Computational and Applied Mathematics, Rice University, Houston, TX, United States
[b] Institute for Computational Engineering and Sciences, The University of Texas at Austin, TX, United States

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ABSTRACT

The Discontinuous Petrov–Galerkin (DPG) method is a class of novel higher order adaptive finite element methods derived from the minimization of the residual of the variational problem (Demkowicz and Gopalakrishnan, 2011) [1], and has been shown to deliver a method for convection–diffusion that is provably robust in the diffusion parameter (Demkowicz and Heuer, in press; Chan et al., in press) [2,3]. In this work, the DPG method is extrapolated to nonlinear systems, and applied to several problems in fluid dynamics whose solutions exhibit boundary layers or singularities in stresses. In particular, the effectiveness of DPG as a numerical method for compressible flow is assessed with the application of DPG to two model problems over a range of Mach numbers and laminar Reynolds numbers using automatic adaptivity with higher order finite elements, beginning with highly under-resolved coarse initial meshes.

1. Introduction

Standard numerical methods tend to perform poorly across the board for the class of PDEs known as singular perturbation problems; these problems are often characterized by a parameter that may be either very small or very large. An additional complication of singular perturbation problems is that very often, in the limiting case of the parameter blowing up or decreasing to zero, the PDE itself will change types (e.g. from elliptic to hyperbolic). A canonical example of a singularly perturbed problem is the convection–diffusion equation in domain $\Omega \subset \mathbb{R}^3$,

$$\nabla \cdot (\beta u) - \epsilon \Delta u = f.$$

The equation models the steady-state distribution of the scalar quantity u, representing the concentration of a quantity in a given medium, taking into account both convective and diffusive effects. Vector $\beta \in \mathbb{R}^3$ specifies the direction and magnitude of convection, while the singular perturbation parameter ϵ represents the diffusivity of the medium. In the limit of an inviscid medium as $\epsilon \to 0$, the equation changes types, from elliptic to hyperbolic, and from second order to first order.

The standard finite element method applied to the convection-dominated diffusion problem can perform very poorly for the case of small ϵ. [1] This poor performance can be observed numerically – as the singular perturbation parameter ϵ decreases, the finite element solution can diverge significantly from the best finite element approximation of the solution. For example, it is well known that, on a fixed coarse mesh and for small values of ϵ (or a large Peclet number, the ratio h/ϵ), the Galerkin approximation of the solution to the convection–diffusion equation with a boundary layer develops spurious oscillations everywhere in the domain, even where the best approximation error is small. These oscillations grow in magnitude as $\epsilon \to 0$, eventually polluting the entire solution.[2]

Traditionally, instability/loss of robustness in finite element methods has been dealt with using residual-based stabilization techniques. Given some variational form, the problem is modified by adding to the bilinear form the strong form of the residual, weighted by a test function and scaled by a stabilization constant τ. The most well-known example of this technique is the streamline-upwind Petrov–Galerkin (SUPG) method, which is a stabilized FE method for solving the convection–diffusion equation using piecewise linear continuous finite elements [5]. SUPG stabilization not only removes the spurious oscillations from the finite element solution of the convection–diffusion equation, but delivers the best finite element approximation in the H^1 norm in 1D.

[1] This is especially true in the presence of boundary layers in the solution [4].

[2] For nonlinear shock problems, the solution often exhibits sharp gradients or discontinuities, around which the solution would develop spurious Gibbs-type oscillations. These are a result of underresolution of the solution, and are separate from the oscillations resulting from a lack of robustness.

* Corresponding author. Tel.: +1 6269935196.
E-mail addresses: jesse.chan@rice.edu (J. Chan), leszek@ices.utexas.edu (L. Demkowicz), rmoser@ices.utexas.edu (R. Moser).

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From the perspective of the compressible Navier–Stokes equations, this loss of robustness is doubly problematic. Not only will any nonlinear solution suffer from similar unstable oscillations, but nonlinear solvers themselves may fail to yield a solution due to such instabilities. A nonlinear solution is almost always computed by solving a series of linear problems whose solutions will converge to the nonlinear solution under appropriate assumptions, and the presence of such oscillations in each linear problem can cause the solution convergence to slow significantly or even diverge. Artificial viscosity and shock capturing methods have been used to suppress such oscillations and regularize the problem. While these methods will usually yield smooth and qualitatively resolved solutions, these methods are often overly diffusive, yielding results which are poor approximations of the true solution [6], though modern artificial viscosity and shock capturing schemes have improved greatly in recent years [7,8]. We have taken an alternative approach in this work, avoiding artificial diffusion and shock capturing for the moment.

Our aim is to develop a stable, higher order scheme for the steady compressible laminar Navier–Stokes equations in transonic/supersonic regimes that is automatically adaptive beginning with very coarse meshes. This requires that both the method and the refinement scheme to perform adequately on coarse meshes with high Peclet numbers – in other words, that the adaptive method is robust in the diffusion parameter. We construct such a method in this paper as follows: we begin by deriving the DPG method for linear problems, then constructing a DPG method for the scalar convection–diffusion equation that is robust for very small viscosities. Unlike common adaptive methods in computational fluid dynamics, which often refine based on physical features (such as high gradients in the solution), adaptivity under this method is driven by the minimization of a residual, which measures error accurately even for highly underresolved meshes. The DPG method for scalar convection–diffusion is then extrapolated to systems of nonlinear equations, and applied to the compressible Navier–Stokes equations. Numerical experiments are given, demonstrating the robustness of the method and the effectiveness of automatic adaptivity for two model problems in viscous compressible flow.

2. DPG as a minimum-residual method

Our starting point is the minimization of some measure of error over a finite-dimensional space, given an abstract variational formulation

$$\begin{cases} \text{Given } l \in V^*, & \text{find } u \in U \text{ such that} \\ b(u,v) = l(v), & \forall v \in V, \end{cases} \tag{1}$$

where $b(\cdot,\cdot) : U \times V \to \mathbb{R}$ is a continuous bilinear form. Throughout the paper, we assume that the trial space U and test space V are real Hilbert spaces, and denote U^* and V^* as the respective topological dual spaces. Throughout the paper, we suppose the variational problem (1) to be well-posed in the inf-sup sense. We can then identify a unique operator $B : U \to V^*$ such that

$$\langle Bu, v\rangle_{V^* \times V} := b(u,v), \quad u \in U, v \in V$$

with $\langle\cdot,\cdot\rangle_{V^*\times V}$ denoting the duality pairing between V^* and V, to obtain the operator form of the our variational problem

$$Bu = l \quad \text{in } V^*. \tag{2}$$

We are interested in minimizing the residual over the discrete approximating subspace $U_h \subset U$

$$u_h = \arg\min_{u_h \in U_h} J(u_h) := \frac{1}{2}\|l - Bu_h\|^2_{V^*} := \frac{1}{2}\sup_{v \in V\setminus\{0\}} \frac{|l(v) - b(u_h,v)|^2}{\|v\|^2_V}.$$

For convenience in writing, we will abuse the notation $\sup_{v\in V}$ to denote $\sup_{v\in V\setminus\{0\}}$ for the remainder of the paper. If we define the problem-dependent *energy norm*

$$\|u\|_E := \|Bu\|_{V^*},$$

then we can equate the minimization of $J(u_h)$ with the minimization of error in $\|\cdot\|_E$.

The first order optimality condition for minimization of the quadratic functional $J(u_h)$ requires the Gâteaux derivative to be zero in all directions $\delta u \in U_h$,

$$(l - Bu_h, B\delta u)_{V^*} = 0, \quad \forall \delta u \in U, \tag{3}$$

which is nothing more than the least-squares condition enforcing orthogonality of error with respect to the inner product on V.

The difficulty in working with the first-order optimality condition (3) is that the inner product $(\cdot,\cdot)_{V^*}$ cannot be evaluated explicitly. However, we have that

$$(l - Bu_h, B\delta u)_{V^*} = \left(R_V^{-1}(l - Bu_h), R_V^{-1}B\delta u\right)_V, \tag{4}$$

where $R_V : V \to V^*$ is the Riesz map mapping elements of a Hilbert space V to elements of the dual V^* defined by

$$\langle R_V v, \delta v\rangle_{V^*\times V} := (v, \delta v)_V.$$

Furthermore, the Riesz operator is an isometry, such that $J(u_h) = \frac{1}{2}\|l - Bu_h\|^2_{V^*} = \frac{1}{2}\left\|R_V^{-1}(l - Bu_h)\right\|^2_V$. Thus, satisfaction of (4) is exactly equivalent to satisfaction of the original optimality conditions (3).

2.1. Optimal test functions

We define, for a given trial function $\delta u \in U$, the corresponding *optimal test function* $v_{\delta u}$

$$v_{\delta u} := R_V^{-1}B\delta u \quad \text{in } V. \tag{5}$$

By definition of the Riesz inverse, the optimality condition (4) becomes

$$\langle Bu_h - l, v_{\delta u}\rangle_V = 0, \quad \forall \delta u \in U$$

which is exactly the standard variational equation in (1) with $v_{\delta u}$ as the test functions. By defining the optimal test space $V_{\text{opt}} := \{v_{\delta u} \text{ s.t. } \delta u \in U\}$, the solution of the discrete variational problem (1) with test space $V_h = V_{\text{opt}}$ minimizes the residual in the dual norm $\|Bu_h - l\|_{V^*}$.

The inversion of the Riesz operator required to determine optimal test functions is done through the solution of the auxiliary variational problem

$$(v_{\delta u}, \delta v)_V = b(\delta u, \delta v), \quad \forall \delta v \in V. \tag{6}$$

In general, for conforming finite element methods, test functions are continuous over the entire domain, and hence solving variational problem (6) for each optimal test function is a global operation over the entire mesh, rendering the method impractical. The development of discontinuous Galerkin (DG) methods allows us to avoid this by adopting basis functions which are discontinuous over elements. In particular, the use of discontinuous test functions δv and a *localizable* norm $\|v\|^2_{V(\Omega_h)} = \sum_{K\in\Omega_h}\|v\|^2_{V(K)}$, (where $\|v\|_{V(K)}$ is a norm over the element K), reduces the problem of determining global optimal test functions in (6) to local problems that can be solved in an element-by-element fashion.

We note that solving (6) on each element exactly is still impossible since it amounts to inverting the infinite-dimensional Riesz map R_V. Instead, optimal test functions are approximated using the standard Bubnov–Galerkin method on an "enriched" subspace $\widetilde{V} \subset V$ such that $\dim(\widetilde{V}) > \dim(U_h)$ elementwise [9,1]. We assume