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A high-order hybrid finite difference–finite volume approach with application to inviscid compressible flow problems: A preliminary study

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ABSTRACT

A class of hybrid finite difference–finite volume (FD–FV) operators is recently developed as building blocks to solve one dimensional hyperbolic conservation laws when the solutions are smooth. This method differs from conventional finite difference (FD) or finite volume (FV) schemes in that both nodal values and cell-averaged values are considered as dependent variables and they are evolved in time. Under this framework, the 1D FD–FV methods: (1) are numerically conservative for cell averages; (2) have straightforward extension to high-order accuracy; and (3) have superior spatial accuracy property compared to most conventional FD or FV methods.

This work extends the FD–FV approach in two aspects. The first extension is a WENO-type stabilization to enhance the nonlinear stability of sample high-order 1D FD–FV operators. In particular, numerical results show that when the solutions are smooth, the optimal order of accuracy (fifth-order) is achieved by the stabilized fourth-order FD–FV method; and it is also capable to handle problems with strongly discontinuous solutions.

The second part of the paper extends a second-order FD–FV method to two-dimensional smooth problems. Both Cartesian grids and unstructured (triangular) grids are considered. In multiple dimensions, there are different choices of the collocation points of the nodal values, and they lead to different FD–FV schemes. This work develops a node-centered FD–FV scheme and an edge-centered FD–FV scheme on each type of grids, and their numerical performance are assessed and compared by solving benchmark flow problems with smooth solutions. In particular, the numerical examples confirm that the superior spatial accuracy property of the 1D FD–FV operators carries to two space dimensions on Cartesian grids. The present work focuses on two space dimensions, but the methodology extends naturally to three-dimensional Cartesian grids and tetrahedral grids.

1. Introduction

The hybrid finite difference–finite volume (FD–FV) operator is recently proposed [1] to solve general one-dimensional (1D) hyperbolic conservation laws when the solutions are smooth. It uses both cell averages and nodal values as dependent variables and use method of lines to evolve these variables in time. In particular, the semi-discretization formula for the cell-averaged variables is obtained directly from the weak form of the governing equation, which leads to inherent conservation for these variables. Hermite interpolation polynomials are used to construct the semi-discretization formula for the nodal values, thus extension to arbitrary high-order accuracy is straightforward.

Numerical schemes for hyperbolic problems of this type date back to the 1970s, and van Leer [2] constructed a third-order accurate method (Scheme V of the reference) using both cell averages and nodal values at cell faces for 1D conservation laws. This work, however, is not further explored. Other similar works in literature are the multi-moment methods [3,4], the staggered mesh or dual mesh approaches [5,6], the space–time control element and solution element methods (CE/SE) [7–9], the P_NP_M methods [10,11], and more recently the active flux schemes [12]. The major difference between the FD–FV approach and these methods is in the following aspects: (1) it has very simple and very general formulation, and does not require any Riemann solvers and (2) it uses method of lines for integration in time, and can be constructed to high-order accuracy in both space and in time.

Another important feature that distinguishes the FD–FV approach from other numerical methods is its superior accuracy property for smooth problems. In particular, it is theoretically

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shown in previous work [1] that in one space dimension, the formal spatial order of accuracy of a FD–FV scheme is one-order higher than the designed order of the discrete differential operator that is used to semi-discretize the equation. In contrast, these two numbers are typically the same for conventional finite difference or finite volume methods. To make this point clear, a case study is provided in Section 3; and this property is formalized in Theorem 4.3. This superior accuracy property in space makes the FD–FV operators attractive building blocks towards constructing efficient solvers for practical hyperbolic problems.

The present work extends this methodology in two aspects. First, since the construction of the 1D FD–FV operators is based on Taylor series expansion, these operators are not expected to perform well when the solutions are discontinuous. Numerical examples show that when the stencil of the operator exceeds one cell in either the upwind or the downwind direction, significant oscillations appear the shock fronts. To this end, the weighted essentially nonoscillatory (WENO) strategy is adopted to enhance the nonlinear stability of high-order 1D FD–FV operators. In particular, a formally third-order accurate FD–FV operator is constructed as a convex combination of two third-order upwind or upwind-biased operators; this WENO-type operator shifts the weight towards the stencil in which the data is smoother, and achieves optimal fourth-order accuracy when the solutions are sufficiently smooth. Note that the order of the FD–FV scheme is one-order higher than the operator, thus the optimal order of spatial accuracy for this WENO-stabilized FD–FV method is fifth-order.

The second part of this work explores the extension of a second-order FD–FV method to two space dimensions. The discussion, however, is limited to smooth problems since the main focus is to investigate the superior accuracy property of the FD–FV framework in multiple space dimensions. In particular, the proposed second-order FD–FV methods are constructed using first-order discrete differential operators, like in the 1D case.

The extension is nontrivial, however, even on structured meshes due to the following reason. The weak formulation over one cell gives rise to surface integrals of the flux functions along the cell boundaries. In 1D, these boundary integrals reduce to point-wise values that are given by flux functions evaluated using the nodal dependent variables. In multiple dimensions, however, these integrals must be approximated numerically, which leads to multiple choices of the collocation points of nodal variables. This also brings another issue that higher-order schemes require more nodal variables on the cell boundaries than low-order schemes. To simplify these issues, this work focuses on constructing second-order FD–FV methods for 2D hyperbolic conservation laws. In this case, second-order approximation of the cell boundary integral is sufficient. To this end, two different configurations of the nodal dependent variables are considered: the first configuration has nodal values collocated at the grid points; and the second one chooses nodal variables at edge centers. FD–FV schemes using both configurations are constructed on both the Cartesian mesh and the unstructured (triangular) mesh. And the accuracy of these methods are assessed by solving a number of benchmark smooth flow problems.

Finally, it is emphasized that the proposed 2D FD–FV schemes do not guarantee nonlinear stability and they are not suitable for problems with discontinuities. Thus the methods presented in this work should be considered as building blocks to construct more practical solvers with enhanced nonlinear stability, which is left for future work. This point is further discussed at the end of Section 5.

The remainder of the paper is organized as follows. Section 2 summarizes the notations used in this work. Then Section 3 provides a simple case study highlighting the main features of the 1D FD–FV operators, and its numerical performance and computational cost is compared to those of two simple finite difference methods. The general 1D FD–FV approach and the fundamental results concerning its accuracy and linear stability are briefly reviewed at the beginning of Section 4. This section also describes a WENO-type stabilization of the third-order FD–FV operators and assesses its performance by solving a number of benchmark flow problems. Section 5 focuses on the 2D extension of a second-order FD–FV scheme to both Cartesian grids and triangular grids, in assumption that the solutions are sufficiently smooth. Their numerical performances are assessed and compared in Section 6. Finally, Section 7 concludes this paper and discusses future directions.

2. Nomenclature

The following notations are used throughout this paper.

$x, y, u, \ldots$	Scalar variables
$\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{f}, \ldots$	Vector-valued variables
$\dagger(\boldsymbol{x}, t)$	Exact value of a variable †
$\bar{\dagger}$	Cell-averaged value of a variable †
t	Time coordinate
x, y	Cartesian coordinates
h	Reference mesh size in space
λ	Courant numbers
$\mathcal{D}\dagger$	A typical FD operator applied to †
$[\mathcal{D}\dagger]$	A typical FD–FV operator applied to †
Euler equations	
u, v	Velocity components
ρ, p	Density and pressure
E	Total energy
γ	Heat capacity ratio
Subscript	
i, j	Mesh grid points' indices on Cartesian meshes
i	Mesh grid point's index on triangular meshes
x, y	Partial derivatives in space
t	Partial derivative in time
Superscript	
$\star$	Exact counterpart of dependent variable

3. Case study: comparison between a simple FD–FV method and a simple FD method

The most important features of the FD–FV approach can be illustrated by solving the scalar advection problem with smooth solution

$$u_t + u_x = 0, \quad x \in [0,\ 4] \tag{3.1}$$

with periodic boundary condition and initial data

$$u(x,0) = u_0(x) = \begin{cases} 1+\cos(\pi x) & 1 \leqslant x \leqslant 3 \\ 0 & x < 1,\ \text{or } x > 3 \end{cases} \tag{3.2}$$

The simplest FD–FV method and several FD schemes are detailed and their numerical performances are compared. In particular, these classical methods include a first-order upwind FD method and a second-order upwind FD method.

In particular, supposing uniform meshes for simplicity, the classical first-order upwind FD method has dependent variables $w_i^n \approx u(ih, n\Delta t),\ i = 0, \ldots, N$ with $h = 4/N$ as illustrated in Fig. 3.1a. The initial condition is given by

$$w_i^0 = u_0(ih), \quad i = 0, \ldots, N \tag{3.3}$$