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Computation of flows with shocks using the Spectral Difference method with artificial viscosity, I: Basic formulation and application

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ABSTRACT

The present work combines the Spectral Difference method with an artificial viscosity based approach to enable high-order computation of compressible fluid flows with discontinuities. The study uses an artificial viscosity approach similar to the high-wavenumber biased artificial viscosity approach (Cook and Cabot, 2005, 2004; Kawai and Lele, 2008) [1–3], extended to an unstructured grid setup. The model employs a bulk viscosity for treating shocks, a shear viscosity for treating turbulence, and an artificial conductivity to handle contact discontinuities. The high-wavenumber biased viscosity is found to stabilize numerical calculations and reduce oscillations near discontinuities. Promising results are demonstrated for 1D and 2D test problems.

1. Introduction

Until recently, compressible flow computations on unstructured meshes have generally been dominated by schemes restricted to second order accuracy. However, the need for highly accurate methods in applications such as large eddy simulation, direct numerical simulation and computational aeroacoustics, has seen the development of higher order schemes for unstructured meshes. In particular, there has been a rise in the popularity and application of locally discontinuous formulations. Methods such as Discontinuous Galerkin (DG) method [4,5], Spectral Volume (SV) method [6,7] and Spectral Difference (SD) method [8,9], Lifting Collocation Penalty (LCP) approach [10], etc. fall under this category.

The SD method is a high-order approach based on the differential form of the conservative equations. This method combines elements from Finite-Volume and Finite-Difference techniques and is particularly attractive because it is conservative, has a simple formulation and straightforward implementation. The absence of volume or surface integrals also makes this method efficient. The origins of the SD method can be traced back to 1996, when Kopriva and Kolias [11] and Kopriva [12] introduced their formulation for the solution of the 2D compressible Euler equations on unstructured quadrilateral meshes, which they called the 'Conservative Staggered-Grid Chebyshev Multi-Domain method'. Liu et al. [8] developed a general formulation of this approach on simplex cells and applied it to wave equations on triangular grids. Wang et al. [9] extended it to 2D Euler equations on triangular grids. It was further extended to the 2D N–S equations by May and Jameson [13], and Wang et al. [14]. Sun et al. [15] further developed it for three-dimensional Navier–Stokes equations on hexahedral unstructured meshes. Recently, Jameson [16] obtained a theoretical proof that the SD method is stable for all orders of accuracy in a Sobolev norm provided that the interior flux points are located at the zeros of the corresponding Legendre polynomial. This is valid for the 1D formulation and applies to tensor-product based quadrilateral and hexahedral cells. However, the SD scheme is not stable on simplex elements. In this regard, Balan et al. [17] proposed an alternate formulation of the SD scheme, featuring a flux interpolation technique using Raviart–Thomas spaces, which exhibits linear stability for triangular elements.

One of the greatest challenges with using high-order methods is their inability to handle flow discontinuities. When flows involve steep gradients such as shock waves or contact surfaces, non-physical spurious oscillations arise that contaminate the solution in smooth regions of the flow often causing the simulations to go unstable. Higher order approximations are less dissipative than their low-order counterparts, and hence it is typically necessary to add explicit dissipation in order to obtain a stable solution. However this has a negative effect on accuracy in the vicinity of the discontinuity. It may also degrade the resolution of turbulent scales due to excessive damping. The development of numerical algorithms that can capture discontinuities and also resolve the scales of turbulence in compressible turbulent flows remains a significant challenge.

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A classical approach to shock capturing is the addition of artificial viscosity (AV), pioneered by von Neumann and Richtmeyer [18]. The concept of flexible addition of artificial viscosity/dissipation has been used very successfully by Jameson et al. [19–22], thus producing non-oscillatory and sharp resolution of shocks for structured and unstructured finite volume calculations. Cook and Cabot proposed such a method for high-order centered differencing schemes, wherein a spectral-like high-wavenumber biased artificial viscosity and diffusivity were dynamically added [1,2]. This was followed up with work by Fiorina and Lele [23], on high-order compact difference schemes, wherein artificial diffusivity was added in addition to artificial viscosity. Kawai and Lele [3] extended the method to non-uniform and curvilinear meshes. This method involves the dynamic addition of grid-dependent localized transport coefficients such as artificial bulk viscosity, shear viscosity and artificial conductivity where needed. This facilitates the capturing of discontinuities by smearing the discontinuity over a numerically resolvable scale. The application of this form of artificial viscosity (hyperviscosity) has been limited to structured grid computations.

Other forms of artificial viscosity have been applied to high-order unstructured grid calculations. Persson and Peraire [24] introduced a p-dependent artificial viscosity and demonstrated that higher-order representations and a piecewise-constant artificial viscosity can be combined to produce sub-cell shock resolution. Barter and Darmofal [25] proposed shock-capturing using a combination of higher-order PDE-based artificial viscosity and enthalpy-preserving dissipation operator. The above methods were proposed for high-order Discontinuous Galerkin (DG) discretizations. Nguyen and Peraire [26] proposed an adaptive shock-capturing approach for the hybridizable DG method. Yang and Wang [27] suggested the use of limiters with SD schemes for shock capturing but reported issues with convergence when using limiters.

The current study focuses on extending the artificial viscosity approach proposed by Cook and Cabot [1,2], and modified by Kawai and Lele [3] to computations on unstructured quadrilateral grids using the Spectral Difference scheme. It must be mentioned that the present manuscript is an extended version of the work submitted to the 2009 AIAA CFD Conference [28]. This paper will discuss the salient aspects of implementing artificial viscosity within the Spectral Difference setup. The applicability and limitations of this approach will be demonstrated with test cases in 1D and 2D. The current implementation of artificial viscosity can also be extended to the 3D Spectral Difference scheme.

In Section 2, we look at the formulation of the SD method on unstructured quadrilateral meshes. Section 3 discusses the details of the artificial viscosity method used. In Section 4, we look at the numerical results obtained from the application of the artificial viscosity method to multiple test cases. Section 5 discusses the conclusions of our study and the direction of future efforts.

2. Formulation of 2D Spectral Difference scheme on quadrilateral meshes

The formulation of the equations for the 2D SD scheme on quadrilateral meshes is similar to the formulation of Sun et al. [15] for unstructured hexahedral grids.

Consider the unsteady compressible 2D Navier Stokes equations in conservative form

$$\frac{\partial Q}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=0 \tag{1}$$

where Q is the vector of conserved variables; F and G are the total flux vectors in the x and y direction respectively. F and G can be split into inviscid and viscous parts, $F=F_i+F_v$ and $G=G_i+G_v$.

The conservative variables and the inviscid components of the fluxes are given by,

$$Q=\begin{pmatrix}\rho\\ \rho u\\ \rho v\\ \rho E\end{pmatrix} F_i=\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho uv\\ u(\rho E+p)\end{pmatrix} G_i=\begin{pmatrix}\rho v\\ \rho uv\\ \rho v^2+p\\ v(\rho E+p)\end{pmatrix} \tag{2}$$

where ρ is the density of the fluid, u and v are the cartesian velocity components of the flow, p is the pressure, and E is the specific total energy.

The viscous flux vectors can be written as

$$F_v=-\begin{pmatrix}0\\ \tau_{xx}\\ \tau_{yx}\\ u\tau_{xx}+v\tau_{yx}+\kappa\frac{\partial T}{\partial x}\end{pmatrix} \tag{3}$$

$$G_v=-\begin{pmatrix}0\\ \tau_{xy}\\ \tau_{yy}\\ u\tau_{xy}+v\tau_{yy}+\kappa\frac{\partial T}{\partial y}\end{pmatrix}$$

where the τ's are components of the shear stress tensor, and κ is the thermal conductivity of the fluid. The shear stress tensor is related to the velocity gradients as given below.

$$\begin{aligned}\tau_{xx}&=2\mu\frac{\partial u}{\partial x}+\beta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\\ \tau_{yy}&=2\mu\frac{\partial v}{\partial y}+\beta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\\ \tau_{yx}&=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\\ \tau_{xy}&=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\end{aligned} \tag{4}$$

where μ is the dynamic (shear) viscosity coefficient, and β is the bulk viscosity coefficient. The latter is related to the viscous stress caused by a volume change. However, under the Stokes' hypothesis, the bulk viscosity is related to the dynamic viscosity as $\beta=-(2/3)\mu$, and the trace of the shear stress tensor vanishes.

To achieve an efficient implementation, all elements in the physical domain (x,y) are transformed into a standard square element, $0<\xi<1,\ 0<\eta<1$. The transformation can be written as:

$$\begin{pmatrix}x\\ y\end{pmatrix}=\sum_{i=1}^{K}M_i(\xi,\eta)\begin{pmatrix}x_i\\ y_i\end{pmatrix} \tag{5}$$

where K is the number of points used to define the physical element, (x_i,y_i) are the cartesian coordinates at those points, and $M_i(\xi,\eta)$ are the shape functions. The metrics and the Jacobian of the transformation can be computed for the standard element. The governing equations in the physical domain are then transferred into the computational domain, and the transformed equations take the following form:

$$\frac{\partial\widetilde{Q}}{\partial t}+\frac{\partial\widetilde{F}}{\partial\xi}+\frac{\partial\widetilde{G}}{\partial\eta}=0 \tag{6}$$

where $\widetilde{Q}=|J|\cdot Q$ and

$$\begin{pmatrix}\widetilde{F}\\ \widetilde{G}\end{pmatrix}=|J|\begin{pmatrix}\xi_x\xi_y\\ \eta_x\eta_y\end{pmatrix}\begin{pmatrix}F\\ G\end{pmatrix} \tag{7}$$

In the standard element, two sets of points are defined, namely the solution points and the flux points, illustrated in Fig. 1. In order to construct a degree $(N-1)$ polynomial in each coordinate direction, the solution at N points are required. The solution points in 1D are chosen to be the Gauss points defined by: