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Devising **HDG** methods for Stokes flow: An overview

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ABSTRACT

We provide a short overview of our recent work on the devising of hybridizable discontinuous Galerkin (**HDG**) methods for the Stokes equations of incompressible flow. First, we motivate and display the general form of the methods and show that they provide a well defined approximate solution for arbitrary polyhedral elements. We then discuss three different but equivalent formulations of the methods. Next, we describe a systematic way of constructing superconvergent **HDG** methods by using, as building blocks, the local spaces of superconvergent **HDG** methods for the Laplacian operator. This can be done, so far, for simplexes, parallelepipeds and prisms. Finally, we show how, by means of an elementwise computation, we can obtain divergence-free velocity approximations converging faster than the original velocity approximation when working with simplicial elements. We end by briefly discussing other versions of the methods, how to obtain **HDG** methods with **H**(div)-conforming velocity spaces, and how to extend the methods to other related systems. Several open problems are described.

1. Introduction

In this paper, we give a short overview of our recent work on the devising of hybridizable discontinuous Galerkin (**HDG**) methods for the velocity gradient–velocity–pressure formulation of the Stokes equations, namely,

$$L - \nabla \boldsymbol{u} = 0 \quad \text{on} \quad \Omega, \tag{1a}$$

$$-\nabla \cdot (\nu L) + \nabla p = \boldsymbol{f} \quad \text{on} \quad \Omega, \tag{1b}$$

$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{on} \quad \Omega, \tag{1c}$$

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \partial\Omega, \tag{1d}$$

$$\int_\Omega p = 0, \tag{1e}$$

where $\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} = 0$. Here $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ is a bounded polygonal domain if $n = 2$, and a Lipschitz polyhedral domain if $n = 3$. We assume that ν is a constant and that f is smooth.

The paper is organized as follows. In Section 2, we begin by describing the characterization of the exact solution the **HDG** methods are obtained from. In Section 3, we use this characterization to define the methods and display very simple conditions that, for elements of arbitrary shape, ensure the existence and uniqueness of their solution. Next, in Section 4, we provide three different ways to presenting the methods according to which unknowns are considered independent and which ones dependent. Then, in Section 5, a fairly general construction of *superconvergent* methods in terms of superconvergent methods for the Laplace operator is presented. These are methods for which, roughly speaking, an elementwise post-processing of the velocity can be obtained which converges faster than the original approximation. Finally, in Section 6, restricting ourselves to simplicial elements, we show how the above-mentioned postprocessing can be defined which results in a globally divergence-free approximate velocity converging faster that the original approximation. We end in Section 7 by briefly considering other versions of the methods, by discussing a new way of obtaining method using $\mathbf{H}(div)$-conforming velocity spaces, and by commenting on how to extend the methods to other related systems.

2. The main idea for devising HDG methods

In this Section, we introduce a characterization of the exact solution whose discrete version gives rise to the **HDG** methods. Given any mesh $\mathcal{T}_h$, which, for simplicity we take to be conforming, of the domain Ω, the characterization we seek states, roughly speaking, that the exact solution solves local Stokes problems which are suitably matched across inter-element boundaries. To find it, we begin with a simple observation.

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2.1. A simple observation

Note that the exact solution satisfies the partial differential equations

$$L - \nabla \boldsymbol{u} = 0 \quad \text{on} \quad K,$$
$$-\nabla \cdot (\nu L) + \nabla p = \boldsymbol{f} \quad \text{on} \quad K,$$
$$\nabla \cdot \boldsymbol{u} = 0 \quad \text{on} \quad K,$$

on each of the elements K of the mesh $\mathcal{T}_h$. Moreover, it satisfies the transmission conditions

$$[\![-\nu L\boldsymbol{n} + p\boldsymbol{n}]\!] = 0 \quad \text{on} \quad F,$$
$$[\![\boldsymbol{u} \otimes \boldsymbol{n}]\!] = 0 \quad \text{on} \quad F,$$

for all the faces F of each of the elements $K \in \mathcal{T}_h$. Here, $[\![\cdot]\!]$ on F denotes the jump across the inter-element boundary F, that is

$$[\![-\nu L\boldsymbol{n} + p\boldsymbol{n}]\!] := -\nu L^- \boldsymbol{n}^- + p^- \boldsymbol{n}^- - \nu L^+ \boldsymbol{n}^+ + p^+ \boldsymbol{n}^+,$$
$$[\![\boldsymbol{u} \otimes \boldsymbol{n}]\!] := \boldsymbol{u}^- \otimes \boldsymbol{n}^- + \boldsymbol{u}^+ \otimes \boldsymbol{n}^+,$$

where $\zeta^\pm$ is the trace on the face F of the generic function ζ from either of its sides. Finally, it satisfies Dirichlet boundary and global average conditions

$$\boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \partial\Omega, \quad \int_\Omega p = 0.$$

Conversely, any function $(L, \boldsymbol{u}, p)$ satisfying the above equations on each of the elements $K \in \mathcal{T}_h$, the transmission conditions on all the faces F of $K \in \mathcal{T}_h$ and the Dirichlet and global average conditions is nothing but the exact solution of the original problem.

2.2. Local and global problems

We are going now to use this simple result to obtain the characterization we seek. We proceed as follows. For an *arbitrary* function $\hat{\boldsymbol{u}}$ defined on the set of all faces F of the elements K of $\mathcal{T}_h$, $\mathcal{E}_h$, and *any* function $\bar{p}$ defined on Ω and constant on each element K of $\mathcal{T}_h$, we define the auxiliary function $(\widetilde{L}, \tilde{\boldsymbol{u}}, \tilde{p})$ as the solution of the *local* problem

$$\widetilde{L} - \nabla \tilde{\boldsymbol{u}} = 0 \quad \text{on} \quad K, \tag{2a}$$
$$-\nabla \cdot (\nu \widetilde{L}) + \nabla \tilde{p} = \boldsymbol{f} \quad \text{on} \quad K, \tag{2b}$$
$$\nabla \cdot \tilde{\boldsymbol{u}} = \frac{1}{|K|} \int_{\partial K} \hat{\boldsymbol{u}} \cdot \boldsymbol{n} \quad \text{on} \quad K, \tag{2c}$$
$$\tilde{\boldsymbol{u}} = \hat{\boldsymbol{u}} \quad \text{on} \quad \partial K, \tag{2d}$$
$$\frac{1}{K} \int_K \tilde{p} = \bar{p}. \tag{2e}$$

Note that the divergence-free condition *has* to be modified for this problem to be solvable for arbitrary functions $\hat{\boldsymbol{u}}$. If we want to keep the equation $\nabla \cdot \tilde{\boldsymbol{u}} = 0$, the function $\hat{\boldsymbol{u}}$ would have to satisfy the compatibility condition $\int_{\partial K} \hat{\boldsymbol{u}} \cdot \hat{\boldsymbol{n}} = 0$.

By the result in the previous subsection, the function $(\hat{\boldsymbol{u}}, \bar{p})$ for which $(\widetilde{L}, \tilde{\boldsymbol{u}}, \tilde{p})$ is nothing but the exact solution of the original problem, $(L, \boldsymbol{u}, p)$, must be the solution of the *global* problem consisting in the transmission condition

$$[\![-\nu \widetilde{L}\boldsymbol{n} + \tilde{p}\boldsymbol{n}]\!] = 0 \quad \text{on} \quad \mathcal{E}_h \setminus \partial\Omega, \tag{3a}$$

the divergence-free condition

$$\int_{\partial K} \hat{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \quad \text{for} \quad K \in \mathcal{T}_h, \tag{3b}$$

and the Dirichlet and global average conditions

$$\hat{\boldsymbol{u}} = \boldsymbol{g} \quad \text{on} \quad \partial\Omega, \tag{3c}$$
$$\int_\Omega \bar{p} = 0. \tag{3d}$$

Note that the second transmission condition, namely $[\![\tilde{\boldsymbol{u}} \otimes \boldsymbol{n}]\!] = 0$ is automatically satisfied since on $\mathcal{E}_h \setminus \partial\Omega$ because $\tilde{\boldsymbol{u}} = \hat{\boldsymbol{u}}$ therein by the boundary condition of the local problems, (2d).

2.3. Characterization of the exact solution

Thus, we have that the exact solution can be characterized as the sum

$$(L, \boldsymbol{u}, p) = (L^{\hat{\boldsymbol{u}}}, \boldsymbol{u}^{\hat{\boldsymbol{u}}}, p^{\hat{\boldsymbol{u}}}) + (L^{\boldsymbol{f}}, \boldsymbol{u}^{\boldsymbol{f}}, p^{\boldsymbol{f}}) + (0, \boldsymbol{0}, \bar{p}),$$

where we denote by $(L^{\hat{\boldsymbol{u}}}, \boldsymbol{u}^{\hat{\boldsymbol{u}}}, p^{\hat{\boldsymbol{u}}})$ the solution $(\widetilde{L}, \tilde{\boldsymbol{u}}, \tilde{p})$ of the local problem (2) with $\boldsymbol{f} := \boldsymbol{0}$ and $\bar{p} := 0$, and by $(L^{\boldsymbol{f}}, \boldsymbol{u}^{\boldsymbol{f}}, p^{\boldsymbol{f}})$ the solution $(\widetilde{L}, \tilde{\boldsymbol{u}}, \tilde{p})$ of the local problem (2) with $\hat{\boldsymbol{u}} := \boldsymbol{0}$ and $\bar{p} := 0$. Note that the solution $(\widetilde{L}, \tilde{\boldsymbol{u}}, \tilde{p})$ of the local problem (2) with $\hat{\boldsymbol{u}} := \boldsymbol{0}$ and $\boldsymbol{f} := \boldsymbol{0}$ is $(0, \boldsymbol{0}, \bar{p})$.

Moreover, the function $(\hat{\boldsymbol{u}}, \bar{p})$ is the solution of the global problem (3) which, given the last identity, we can rewrite as follows:

$$-[\![-\nu L^{\hat{\boldsymbol{u}}}\boldsymbol{n} + p^{\hat{\boldsymbol{u}}}\boldsymbol{n}]\!] - [\![\bar{p}\boldsymbol{n}]\!] = [\![-\nu L^{\boldsymbol{f}}\boldsymbol{n} + p^{\boldsymbol{f}}\boldsymbol{n}]\!] \quad \text{on} \quad \mathcal{E}_h,$$
$$\int_{\partial K} \hat{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \quad \text{for} \quad K \in \mathcal{T}_h,$$
$$\hat{\boldsymbol{u}} = \boldsymbol{g} \quad \text{on} \quad \partial\Omega,$$
$$\int_\Omega \bar{p} = 0.$$

This characterization of the exact solution is convenient for devising numerical methods because any discrete version of it will consist of local problems written in terms of approximations to $(\hat{\boldsymbol{u}}, \bar{p}), (\hat{\boldsymbol{u}}_h, \bar{p}_h)$, and a single global problem for $(\hat{\boldsymbol{u}}_h, \bar{p}_h)$ *only*. This allows for a very efficient implementation of the method.

3. Definition of the HDG methods

In this Section, we introduce **HDG** methods by discretizing the local problems (2) by discontinuous Galerkin methods, and by enforcing the global problem (3) in a weak manner.

3.1. The approximating spaces

The **HDG** methods seek an approximation $(L_h, \boldsymbol{u}_h, p_h)$ to the exact solution $(L|_\Omega, \boldsymbol{u}|_\Omega, p|_\Omega)$ in the finite dimensional space $G_h \times \boldsymbol{V}_h \times Q_h$ given by

$$G_h = \{G \in L^2(\mathcal{T}_h) : \quad G|_K \in G(K) \quad \forall \quad K \in \mathcal{T}_h\}, \tag{4a}$$
$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{T}_h) : \quad \boldsymbol{v}|_K \in \quad \boldsymbol{V}(K) \quad \forall \quad K \in \mathcal{T}_h\}, \tag{4b}$$
$$Q_h = \{q \in L^2(\mathcal{T}_h) : \quad q|_K \in Q(K) \quad \forall \quad K \in \mathcal{T}_h\}, \tag{4c}$$

where the *local* spaces $G(K), \boldsymbol{V}(K), Q(K)$ are general finite dimensional spaces.

The **HDG** methods also seek an approximation $(\hat{\boldsymbol{u}}_h, \bar{p}_h)$ to the exact solution $(\boldsymbol{u}|_{\mathcal{E}_h}, \bar{p})$ in the space $\boldsymbol{M}_h \times Q_h^0$ where

$$Q_h^0 = \{q \in L^2(\mathcal{T}_h) : \quad q|_K \text{is a constant} \quad \forall \quad K \in \mathcal{T}_h\}, \tag{5a}$$
$$\boldsymbol{M}_h = \{\boldsymbol{\mu} \in \boldsymbol{L}^2(\mathcal{E}_h) : \quad \boldsymbol{\mu}|_F \in \boldsymbol{M}(F) \quad \forall \quad F \in \mathcal{E}_h\}, \tag{5b}$$

where the *local* space $\boldsymbol{M}(F)$ is a general finite dimensional space.

3.2. The local and the global problems

Writing $(\zeta, \eta)_K$ for the integral over the element K of $\zeta\eta$, and $\langle \zeta\eta \rangle_{\partial K}$ for the corresponding integral over ∂K, it is not difficult to