Systems & Control Letters 59 (2010) 536–542

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Distributed average consensus via gossip algorithm with real-valued and quantized data for $0 < q < 1$

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ARTICLE INFO

Article history:
Received 11 October 2009
Received in revised form
24 April 2010
Accepted 17 June 2010
Available online 8 August 2010

Keywords:
Average consensus
Gossip algorithm
Probabilistic quantization

ABSTRACT

This paper studies the problem of the gossip consensus algorithm with real-valued and quantized data. We study the effect of the mixing parameter on the convergence rate of the proposed gossip consensus algorithm, and show when the proposed bounds are optimized with respect to the mixing parameter. For a gossip consensus algorithm with quantized data, we prove that it can achieve the consensus almost surely, and the expected value of the final states is equal to the average of the initial states. Moreover, we provide a result characterizing the convergence performance of the distance from consensus and make a comparison with the non-quantized gossip consensus algorithm. Finally, simulation results are provided to evaluate the effectiveness of the proposed algorithm.

1. Introduction

The study of average consensus algorithms has attracted a lot of interest in the past years [1–5]. Average consensus algorithms consist of computing the average of the initial state values in a distributed fashion. Applications of average consensus algorithms can be found in sensor networks [6–8], distributed coordination control of multiple autonomous agents [9–11], and other areas. Among them, one particular algorithm called gossip, which is studied in-depth in [12], has attracted a lot of interest for its appealing features: it distributes the computational burden; it can efficiently avoid data collision and it is robust to the change of network topology, to name a few. Due to the fact that each node's energy storage and computational power may be limited, and the links that connect any two neighboring nodes can be subjected to bandwidth constraints, message quantization should be considered.

1.1. Related work

The quantization effects due to communication constraints have been considered in many recent papers on time-invariant consensus problem [13–19], while little research has been done on the topic of quantization in the context of the time-varying consensus problem, like the gossip algorithm [20–24]. Kashyap et al. start the research on consensus algorithm under quantized communication from a different point of view [20], where the authors require that the network must have and transmit integer-valued data. They have proposed a class of gossip algorithm that the network's average is preserved at each iteration. Furthermore, the notion of quantized consensus has been defined in the sense that the states of the nodes in the network are guaranteed to converge up to one quantization bin. However, the quantized consensus is obviously not a precise consensus; that is, the nodes in the network may have different state values in the end. In [22,23], the authors consider the expected value of the time at which the quantized consensus is achieved; they obtain a small convergence time by solving a convex optimization problem, while we focus on studying the the effect of the mixing parameter on the the rate of convergence to consensus and the proposed bounds. The effect of quantized communication on the gossip algorithm has been considered in [24], where a class of quantized gossip consensus algorithm, named the totally quantized gossip algorithm, has been proposed. In their setting, the mixing parameter is $1/2$. However, our algorithm deals with a more general case; that is, the mixing parameter ranges from 0 to 1.

1.2. Our results

In this paper, we first give a result on gossip consensus algorithm with real-valued data; that is, the data exchanged between any two nodes are non-quantized. We characterize the convergence performance of the proposed algorithm and make a comparison with the standard gossip algorithm [12]. Then, we analyze the effect of quantized communication on the gossip consensus algorithm; that is, we assume the data exchanged

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doi:10.1016/j.sysconle.2010.06.010

between any two nodes are quantized, and show how quantization affects the evolution of the gossip consensus algorithm. We demonstrate that our algorithm indeed achieves the probabilistic average consensus; that is, it reaches the average consensus almost surely. We also provide a result characterizing the convergence performance of the distance from consensus. Furthermore, we derive when the proposed bound is optimized with respect to the mixing parameter. Finally, simulation results are provided to show the effectiveness of the proposed algorithm.

This paper is organized as follows: Section 2 introduces some notations and preliminaries on graph theory and probabilistic quantization, and gives a detailed description of the proposed algorithm. Section 3 describes the gossip consensus algorithm with real-valued data, and contains the main analysis result on this algorithm. Section 4 describes the quantized gossip consensus algorithm and provides performance analysis results. Section 5 illustrates our proposed algorithm through simulation examples. We conclude the paper in Section 6.

2. Problem formulation

In this section, we first review some basic concepts and properties from graph theory, and then we give a brief description of the probabilistic quantization. We also present a lemma which will be useful in the sequel. Finally, we propose our algorithm.

2.1. Graph model

Denote $\mathcal{G} = (V, E)$ as an undirected connected graph, $V = \{1, 2, \dots, N\}$ denotes the set of vertices and $E \subset \{(i, j) : i, j \in V\}$ denotes the set of edges. Denote $\mathcal{N}_i = \{j|(i, j) \in E\}$ the set of neighbors of vertex i and d_i the number of neighbors of the vertex i. Each vertex of the graph is referred to as a node, and endowed with a state $x_i(t)$, for every $i \in V$. The edge $(i, j) \in E$ indicates that node i and node j can establish a bidirectional noise-free communication link with each other. Moreover, we always assume that transmissions are successful.

2.2. Probabilistic quantization

The probabilistic quantization has been introduced in [18]. Following [18], we give a brief review of the probabilistic quantization.

The probabilistic quantization $\mathcal{Q} : \mathbb{R} \to \mathbb{R}$ is defined as follows: suppose $x \in \mathbb{R}$ is bounded to a finite interval $[-I, I]$, and the interval is equally divided into $M - 1$ sub-intervals with quantization points defined by the set $\theta = \{\theta_1, \theta_2, \dots, \theta_M\}$, where $\theta_1 = -I, \theta_M = I$. Denote the interval as $\Delta = \theta_{i+1} - \theta_i$, for $i \in \{1, 2, \dots, M - 1\}$. Then, for $x \in [\theta_i, \theta_{i+1})$, $\mathcal{Q}(x)$ is a random variable defined by

$$\mathcal{Q}(x) = \begin{cases} \theta_i & \text{with probability } (\theta_{i+1} - x)/\Delta \\ \theta_{i+1} & \text{with probability } (x - \theta_i)/\Delta. \end{cases}$$

The following lemma, adopted from [25], gives two important properties of the probabilistic quantizer.

Lemma 1. *For every* $x \in [\theta_i, \theta_{i+1})$,

$$\mathbb{E}\left[\mathcal{Q}(x)\right] = x, \qquad \mathbb{E}\left[(x - \mathcal{Q}(x))^2\right] \le \frac{\Delta^2}{4}. \tag{1}$$

Note that $\mathcal{Q}(x)$ is an unbiased uniform quantizer; that is, the quantized data $\mathcal{Q}(x)$ is an unbiased representation of x.

2.3. Proposed algorithm

We assume at every time instant t the node $i \in V$ is chosen at random with probability $1/N$, and then with probability P_{ij} it contacts one of its neighbors node j such that $\sum_{j\in\mathcal{N}_i} P_{ij} = 1$. We can find that $P_{ji} \neq P_{ij}$ in general, furthermore, we can see that the edge (i, j) is chosen with probability $P_{(i,j)} = \frac{1}{N}(P_{ij} + P_{ji})$. Denote by $P = [P_{ij}]$ the $N \times N$ matrix of nonnegative entries with the condition $P_{ij} \ge 0$ provided that $(i, j) \in E$. We make the same assumption on P as in [12] that P is a stochastic matrix with its largest eigenvalue equal to 1, and all the other $n - 1$ eigenvalues strictly less than 1.

Denote $x(t)$ the vector of state values at the end of the time instant t. Then, the edge (i, j) is selected with probability $P_{(i,j)}$, and the states of node i and node j evolve as follows

$$x_i(t + 1) = (1 - q)x_i(t) + qx_j(t) \tag{2}$$

$$x_j(t + 1) = qx_i(t) + (1 - q)x_j(t) \tag{3}$$

$$x_k(t + 1) = x_k(t) \quad \text{for } k \neq i, j \tag{4}$$

where $q \in (0, 1)$ is called the mixing parameter. Furthermore, it can be compactly written as

$$x(t + 1) = W(t)x(t) \tag{5}$$

where with probability $P_{(i,j)}$ the random matrix $W(t)$ is

$$W^{ij} = I - q(e_i - e_j)(e_i - e_j)^T$$

where $e_i, i = 1, \dots, N$ denotes the column vector in $\mathbb{R}^N$ having all entries equal to 0 except a 1 in the ith position.

For the quantized version, the states evolve according to the following equation

$$x(t + 1) = W(t)\hat{x}(t)$$

where $\hat{x}(t) = \mathcal{Q}(x(t)) = [\mathcal{Q}(x_1(t)), \mathcal{Q}(x_2(t)), \dots, \mathcal{Q}(x_N(t))]^T$, and $W(t)$ is the same as above.

3. Gossip algorithm with real-valued data

In this section, we assume that the data exchanged between any two nodes are real-valued, and the states evolve according to (2)–(4).

In order to derive the convergence of $x(t)$ to average consensus, we investigate the error defined by

$$z(t) = (I - J)x(t)$$

where $J = \frac{1}{N}\mathbf{1}\mathbf{1}^T$ and $\mathbf{1} \in \mathbb{R}^N$ denotes the vector with all its entries equal to 1. Note that $Jx(t) = Jx(0)$; that is, the network's initial state average is preserved (this can be easily seen from (2)–(4)). Moreover, $z(t)$ gives a measure of how far $x(t)$ away from the average consensus. Now, we consider the evolution of $z(t + 1)$

$$\begin{aligned} z(t + 1) &= (I - J)x(t + 1) \\ &= (I - J)W(t)x(t) \\ &= W(t)(I - J)x(t) \\ &= W(t)z(t). \end{aligned}$$

Then, we can write

$$\begin{aligned} \mathbb{E}\left[\|z(t + 1)\|^2 | z(t)\right] &= z(t)^T\mathbb{E}[W(t)^TW(t)]z(t) \\ &\le \lambda_2(\mathbb{E}[W(t)^TW(t)])\|z(t)\|^2. \end{aligned} \tag{6}$$

The last inequality follows from the fact that $z(t)\perp\mathbf{1}$, and $\mathbf{1}$ is the eigenvector corresponding to the largest eigenvalue 1 of $\mathbb{E}[W(t)^TW(t)]$. Now, by repeatedly conditioning and using the iteration obtained above, we have

$$\mathbb{E}[\|z(t)\|^2] \le \lambda_2^t(\mathbb{E}[W(t)^TW(t)])\|z(0)\|^2. \tag{7}$$

It can be seen that the inequality obtained above is the one derived in [12]. However, $W(t)$ here is more general and we'll see in the sequel that it's not a projection matrix in general; that is, $W(t)^TW(t) \neq W(t)$, when $q \neq 1/2$, while $q = 1/2$ is exactly the