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# An extension of LaSalle's Invariance Principle for a class of switched linear systems[☆]

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ABSTRACT

In this paper LaSalle's Invariance Principle for switched linear systems is studied. Unlike most existing results in which each switching mode in the system needs to be asymptotically stable, in this paper the switching modes are allowed to be only Lyapunov stable. Under certain ergodicity assumptions, an extension of LaSalle's Invariance Principle for global asymptotic stability of switched linear systems is proposed provided that the kernels of derivatives of a common quadratic Lyapunov function with respect to the switching modes are disjoint (except the origin).



## 1. Introduction

A switched system can be considered as one of the simplest hybrid systems [1]. It is used for modeling various control problems and some complex processes in nature such as the consensus of multi-agent systems [2,3]. One of the most fundamental problems for switched systems is the stability problem. We refer to [4,5] and the references therein for recent developments. So far, some particular efforts have been put on switched linear systems [6].

In this paper we consider a switched linear system as

$$\dot{x} = A_{\sigma(t)}x, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $\sigma : [0, +\infty) \to \Lambda = \{1, 2, \ldots, N\}$ is a piece-wise constant, right-continuous function, called a switching signal (or switching law).

We all know that for a non-switching linear system asymptotic stability is assured if and only if there exists a quadratic Lyapunov function with a negative definite total derivative. The intuition is that in this case the only "nondecreasing" solution is the origin itself. Motivated by this fact, for switched linear systems it is very natural for us to search for a (common) quadratic Lyapunov function (CQLF). It is obvious that the existence of a CQLF is sufficient for asymptotic stability of the switched system (1.1) under arbitrary switches. Finding a CQLF for a set of Hurwitz matrices is an interesting and challenging problem. There is a large amount of literature concerning it. We refer to [7–10] and the references therein. It is worth mentioning that it was shown in [11] that the existence of a common quadratic Lyapunov function is only a sufficient condition for switched linear systems to be asymptotically stable.

In this paper we are interested in the case where the total derivative of a candidate Lyapunov function with respect to each mode is only non-positive. Such a function is called a weak Lyapunov function [12]. In order to solve the stability problem for this case, various extensions of LaSalle's Invariance Principle for hybrid systems have been investigated [13,14]. By imposing certain observability requirement and restriction on the admissible trajectories, an extended LaSalle's Invariance Principle is obtained for switched linear systems [15]. Then it is extended to switched nonlinear systems [16]. A more traditional style extension of LaSalle's Invariance Principle, which emphasizes on set attraction, is proposed in [12]. Under certain restrictions, another extension of LaSalle's Invariance Principle for switched nonlinear systems and criteria for asymptotic stability is obtained in [17].

So far almost all these extensions of LaSalle's Invariance Principle require each switching mode to be asymptotically stable. Naturally this requirement would be necessary if we do not impose certain restrictions on the switching signals, otherwise when

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the system stays on a non-asymptotically stable mode, the overall system will not be asymptotically stable.

In [18], we considered the case in which some switching modes are allowed to be only Lyapunov stable and imposed some ergodic constraints on the switching signals. By making some assumptions on the dynamics of each mode, we were able to show that all admissible solutions contained in the union of the null space corresponding to the CQLF converge to the origin.

Different from [18], in this paper we propose a stability criterion that is only based on a geometric property of the union of the null space corresponding to the CQLF. Namely, we assume that this set is disjoint except at the origin.

The rest of the paper is organized as follows: Section 2 gives some preliminary knowledge and defines the so-called common joint quadratic Lyapunov function (CJQLF). Certain properties of such functions are also reviewed. In Section 3, an extension of LaSalle's Invariance Principle is proposed, which assures global asymptotic stability of the switched system under certain ergodicity assumptions. Section 4 is the conclusion.

## 2. Preliminaries

Consider a linear system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n. \tag{2.1}$$

Assume there exists a weak quadratic Lyapunov function $V(x) = x^{\mathrm{T}}Px$, with $P > 0$ a positive definite matrix, such that

$$PA + A^{\mathrm{T}}P = -Q, \tag{2.2}$$

where $Q \geq 0$. It is well known that (refer to [15,19]) by LaSalle's Invariance Principle every solution of (2.1) converges to the largest invariant set $M$ contained in $\ker(Q)$, the kernel of $Q$. In fact in this case $M$ is the largest $A$-invariant subspace contained in $\ker(Q)$, which is precisely the unobservable subspace of the pair $(Q, A)$.

Now consider the switched linear system (1.1). It has been pointed out that [12,15] a certain restriction on the switching signals is necessary for extending LaSalle's Invariance Principle. A switching system is said to have a non-vanishing dwell time, if there exists a positive time period $\tau_0 > 0$, such that the switching times $\{\tau_k \mid k = 1, 2, \ldots\}$ satisfy

$$\inf_k(\tau_{k+1} - \tau_k) \geq \tau_0. \tag{2.3}$$

Through this paper we assume that

**A1.** Admissible switching signals have a dwell time $\tau_0 > 0$.

In this paper we explore a new extension, which allows Lyapunov stable modes. Of course, in this case some restrictions on the switching signal is necessary. In addition to Assumption **A1**, we need the following ergodicity assumption on the switching signals.

**A2.** For any $T > 0$, and any $\lambda \in \Lambda$, there exists $t > T$ such that

$$\sigma(t) = \lambda. \tag{2.4}$$

**Remark.** Assumptions A1 and A2 imply that each mode will be active infinite times and the total length for each mode $i$ being active is infinity, i.e.,

$$|\{t \mid \sigma(t) = \lambda\}| = \infty, \quad \forall \lambda \in \Lambda,$$

where $|\cdot|$ denotes the Lebesgue measure.

In the paper our development of an extension of LaSalle's Invariance Principle is based on the following concept that was first proposed by us in [18].

**Definition 2.1.** Consider system (1.1).

1. If a quadratic function $V(x) = x^{\mathrm{T}}Px$ with positive definite $P > 0$ has the following property
$$PA_i + A_i^{\mathrm{T}}P = -Q_i \leq 0, \quad i = 1, \ldots, N, \tag{2.5}$$
then $V(x)$ (or briefly, $P$) is called a common weak quadratic Lyapunov function (CWQLF) of system (1.1).
2. A common weak quadratic Lyapunov function of system (1.1) is called a common joint quadratic Lyapunov function (CJQLF) if
$$Q := \sum_{i=1}^{N} Q_i > 0. \tag{2.6}$$

We have the following property that was first presented in [18] without proof.

**Proposition 2.2.** *For system (1.1), assume there exists a CWQLF $P > 0$, then $P$ is a CJQLF if and only if*

$$\bigcap_{i\in\Lambda} Z_i = \{0\}, \tag{2.7}$$

*where $Z_i$ is the kernel of $Q_i$, $i \in \Lambda$.*

**Proof.** Necessity: if it is not true, there exists a nonzero $\eta \in \bigcap_{i\in\Lambda} Z_i$. Since $Z_i$ is the kernel of $Q_i$, $i \in \Lambda$, then $Q_i\eta = 0$, $\forall\, i \in \Lambda$. Thus $\eta^{\mathrm{T}}Q\eta = \eta^{\mathrm{T}}(\sum_{i\in\Lambda} Q_i)\eta = 0$, which is a contradiction to $Q > 0$.

Sufficiency: $\{0\} = \bigcap_{i\in\Lambda} Z_i = \{x \mid Q_ix = 0, \forall\, i \in \Lambda\}$. Since $Q_i$, $i \in \Lambda$ are semi-positive definite, then $x^{\mathrm{T}}Q_ix = 0$ if and only if $Q_ix = 0$, $i \in \Lambda$.

Obviously, $Q = \sum_{i\in\Lambda} Q_i \geq 0$ and $x^{\mathrm{T}}Qx = \sum_{i\in\Lambda} x^{\mathrm{T}}Q_ix = 0$ if and only if $x = 0$. Therefore, $Q > 0$. The conclusion follows. □

Now, a fundamental question is: under the assumptions of A1 and A2, is the existence of a CJQLF enough to assure global asymptotical stability?

Unfortunately, the answer is negative and [18] gave a counter example.

Therefore, in addition to Assumptions A1, A2, and the existence of a CJQLF, in the following we give some additional conditions such that the system becomes globally asymptotically stable.

## 3. LaSalle's Invariance Principle for disconnected $Z \setminus \{0\}$

Let $Q_i$ $(i \in \Lambda)$ be defined as in (2.5), and assume $\underbrace{0, \ldots, 0}_{n_i}$, $\lambda^i_{n_i+1}, \ldots, \lambda^i_n$ are the eigenvalues of $Q_i$ and $\xi^i_1, \ldots, \xi^i_{n_i}$, $\xi^i_{n_i+1}, \ldots, \xi^i_n$ are the corresponding eigenvectors, where $\lambda^i_j > 0$, $j = n_i + 1, \ldots, n$, $i \in \Lambda$. Without loss of generality, we assume $n_i > 0$. Since $Q_i$ is symmetric, all the eigenvectors $\xi^i_1, \ldots, \xi^i_n$ can be chosen to be linearly independent.

As the kernel of $Q_i$, we denote $Z_i = \mathrm{span}\{\xi^i_1, \ldots, \xi^i_{n_i}\}$, and $H_i = Z_i^{\perp}$ the orthogonal complement of $Z_i$, $i \in \Lambda$.

Now we define a set of cones as

$$C_i = \left\{ z_i + y_i \mid z_i \in Z_i, y_i \in H_i \text{ and } \|y_i\| < \frac{1}{L}\|z_i\| \right\} \subset \mathbb{R}^n, \quad i \in \Lambda, \tag{3.1}$$

where $L > 0$ will be determined later. Note that $C_i$ is an open cone with $x = 0$ as its vertex. Moreover, $C_i$ is an open neighborhood of $Z_i \setminus \{0\}$. (We refer to Fig. 1 for the geometric shape of the cone $C_i$ on the plane.)

Before presenting our main results, we first give the following lemma.

**Lemma 3.1.** *For system (1.1), assume there exists a CJQLF, then there exists $L > 0$ large enough such that $\bigcap_{i\in\Lambda} C_i = \emptyset$, where $C_i$ are defined in (3.1).*