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A note on a class of observability problems for PDEs

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ABSTRACT

The question of observability arises naturally in the analysis of control problems. If the solution of a PDE initial-boundary value problem is known to be zero in a part of the domain, does this guarantee it is zero everywhere? The most popular techniques to establish such results are based on local unique continuation results (Holmgren's theorem) or Carleman estimates. The purpose of this note is to draw attention to a class of problems where the observed region is bounded by characteristics, and local unique continuation fails. Nevertheless, observability may hold. A problem of this nature arose in recent work by the author on control of viscoelastic flows [M. Renardy, Are viscoelastic flows under control or out of control? System Control Lett. 54 (2005) 1183–1193]. In this note, we first analyze a simple example which shares the same essential features. Specifically, we consider the problem $u_{xt} = \alpha u$, for spatially periodic solutions. We show that observability holds for data given on the line $x = 0$. We shall show, however, that there is no observability estimate. We shall then show how the methods used in the more elementary example can be extended to the case of viscoelastic shear flows.

1. Introduction

In the theory of control of partial differential equations, it is of interest whether a system can be controlled by an input that affects only a part of the physical domain or only the boundary. Results along such lines have been established for many types of linear partial differential equations, see e.g. [4,6,12]. Typically, the strategy for proving controllability results is based on "observability" for a dual problem. The question of observability is whether a solution of a PDE is uniquely determined by its values in a subdomain.

The control of viscoelastic media poses some new issues. Early papers on the subject [5,7–10] basically viewed viscoelastic media as a perturbation of the elastic case and obtained results on controllability and observability which are analogous to what is known for equations of hyperbolic type. In this setting, a "state" of the system consists of displacement and velocity, as in the elastic case, and "controllability" refers to control of these variables. In contrast to the elastic case, however, displacement and velocity in a viscoelastic medium are not sufficient to determine the future evolution of the system. The problem changes if the stresses, in addition to the deformation and velocity, are to be controlled. For viscoelastic fluids of Maxwell or Jeffreys type, this question was considered in [2,11]. For the coupled system describing the evolution of velocity and stresses, any spatial boundary is characteristic. This means that local unique continuation results such as Holmgren's theorem do not hold. Nevertheless, it has been shown that observability results can hold. The objective of this note is to study this issue in more detail. Specifically, we shall prove that even though observability holds, there is no observability estimate in any Sobolev norm.

We shall first investigate the issue in a simple example given by the equation $u_{xt} = \alpha u$. This example shares the main features of the viscoelastic problem, i.e. spatial boundaries are characteristic, and there is a family of eigenvalues converging to a finite limit. We shall then show how the same techniques extend to shear flows of viscoelastic fluids.

2. A simple example

We consider solutions of the partial differential equation

$$u_{xt} = \alpha u, \tag{1}$$

where α is a constant. We impose periodic boundary conditions,

$$u(x + 2\pi, t) = u(x, t) \tag{2}$$

and an initial condition at $t = 0$,

$$u(x, 0) = \phi(x). \tag{3}$$

If $\phi(x)$ is given such that

$$\int_0^{2\pi} \phi(x)\,\mathrm{d}x = 0, \tag{4}$$

then this is a well-posed initial value problem. Indeed, if

$$\phi(x) = \sum_{n\neq 0} a_n \exp(inx), \tag{5}$$

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then

$$u(x,t)=\sum_{n\neq 0} a_n \exp(\mathrm{i}nx-\alpha \mathrm{i}t/n). \tag{6}$$

We shall prove the following observability statement.

Theorem 1. *Let α be any nonzero constant. Assume that $\sum |a_n| < \infty$ and that $u(0,t)=0$ for t in some interval (a,b), where $0\le a<b<\infty$. Then $u(x,t)$ is identically zero.*

Proof. It follows from (6) that $u(0,t)$ is an entire function of t, hence $u(0,t)=0$ for all $t>0$. Now consider the Laplace transform

$$F(\lambda)=\int_0^\infty u(0,t)\exp(-\lambda t)\,\mathrm{d}t=\sum_{n\neq 0}\frac{a_n}{\lambda+\mathrm{i}\alpha/n}. \tag{7}$$

Since $u(0,t)=0$, we must of course have $F(\lambda)=0$. On the other hand, the series representation is valid for $\operatorname{Re}\lambda>|\operatorname{Im}\alpha|$, and it defines a function which is analytic except for poles at $-\mathrm{i}\alpha/n$, and a nonisolated singularity at 0. Since F is actually zero, the residues at the poles, i.e. the a_n, must be zero.

The observability theorem we just proved is clearly false if $\alpha=0$, since, in that case, any function of the form $u(x,t)=\chi(x)$, with arbitrary χ, is a solution. Hence, the result depends on the lower order term in the differential equation. It also depends on the periodic boundary condition, as the next result shows. □

Theorem 2. *There exists a solution of $u_{xt}=\alpha u$ such that $u=0$ for $x<0$, but u is not identically zero. Moreover, we can make u of class C^∞ across $x=0$.*

Proof. This is a special case of Theorem 8.6.7 in [3], but for this simple example we can give a more elementary argument.

For a function $w(x)$, define

$$\mathcal{L}w(x)=\int_0^x w(\xi)\,\mathrm{d}\xi. \tag{8}$$

It is then easy to see that any function of the form

$$u(x,t)=\sum_{n=0}^\infty \frac{(\alpha t)^n}{n!}\mathcal{L}^n w(x) \tag{9}$$

is a solution of the partial differential equation. To establish the theorem, simply take w to be C^∞ with support in $[0,\infty)$.

The next theorem is an "anti estimate" result. □

Theorem 3. *Let $0\le a<b<\infty$, $0\le c<d<\infty$ and $0<\epsilon<2\pi$ be given, and let Q be any positive integer. Then there does not exist a constant C such that*

$$\|u\|_{L^2((0,2\pi)\times(a,b))}\le C\|u\|_{H^Q((0,\epsilon)\times(c,d))} \tag{10}$$

for all spatially 2π-periodic solutions of $u_{xt}=\alpha u$.

Proof. Pick any integer M. At $t=0$, we prescribe a smooth initial condition $\phi(x)$ with the properties that $\phi(x)=0$ for $x\in(0,\epsilon)$, ϕ is not identically zero, and

$$\int_0^{2\pi} x^k\phi(x)\,\mathrm{d}x=0 \tag{11}$$

for $k=0,1,\ldots,M$. In terms of the Fourier coefficients, this yields the conditions

$$\sum_{n\neq 0} a_n\exp(\mathrm{i}nx)=0 \tag{12}$$

for $x\in(0,\epsilon)$, $a_0=0$, and

$$\sum_{n\neq 0} n^{-k}a_n=0 \tag{13}$$

for $k=1,\ldots,M$. Let now

$$f_k(x)=\sum_{n\neq 0} n^{-k}a_n\exp(\mathrm{i}nx). \tag{14}$$

Since $\mathrm{d}^k f_k(x)/\mathrm{d}x^k=0$ for $x\in(0,\epsilon)$, and $f_k(0)=f_k'(0)=\cdots=f_k^{(k-1)}(0)=0$, it follows that $f_k(x)=0$ for $x\in(0,\epsilon)$ and $k=0,\ldots,M$.

We have

$$\exp(-\mathrm{i}\alpha t/n)=\sum_{k=0}^\infty \frac{(-\mathrm{i}\alpha t)^k}{k!}n^{-k}, \tag{15}$$

and by using this in the Fourier series (6), we find

$$u(x,t)=\sum_{k=0}^\infty f_k(x)\frac{(-\mathrm{i}\alpha t)^k}{k!}. \tag{16}$$

For $x\in(0,\epsilon)$, we therefore find

$$u(x,t)=\sum_{k=M+1}^\infty \frac{(-\mathrm{i}\alpha t)^k}{k!}f_k(x). \tag{17}$$

We therefore find

$$\begin{aligned}\|u\|_{L^2((0,\epsilon)\times(c,d))}&\le\sum_{k=M+1}^\infty\frac{(d|\alpha|)^k}{k!}\|f_k\|_{L^2(0,\epsilon)}\\&\le\|\phi\|_{L^2(0,2\pi)}\sum_{k=M+1}^\infty\frac{(d|\alpha|)^k}{k!}.\end{aligned} \tag{18}$$

We can make the sum on the right hand side arbitrarily small by choosing M large. Derivatives of u can be bounded in a similar fashion; note that $f_k'(x)=\mathrm{i}f_{k-1}(x)$, and hence, for $x\in(0,\epsilon)$,

$$\begin{aligned}u_x(x,t)&=\sum_{k=M+1}^\infty \mathrm{i}f_{k-1}(x)\frac{(-\mathrm{i}\alpha t)^k}{k!},\\u_t(x,t)&=\sum_{k=M}^\infty -\mathrm{i}\alpha f_{k+1}(x)\frac{(-\mathrm{i}\alpha t)^k}{k!}.\end{aligned} \tag{19}$$

These sums can be analyzed in exactly the same fashion as the sum for u. On the other hand, it is easy to see from (6) that

$$\|u\|_{L^2((0,2\pi)\times(a,b))}\ge C\|\phi\|_{L^2(0,2\pi)} \tag{20}$$

with a constant that depends only on a, b and α. □

3. Viscoelastic shear flows

For simplicity of exposition, we shall stick to a single mode Jeffreys model; it is straightforward to extend similar arguments to multi mode Maxwell or Jeffreys models as studied in [11]. We shall thus consider a problem of the form

$$\alpha u_{tt}+\beta u_t+\gamma u=u_{xxt}+\mu u_{xx}, \tag{21}$$

with boundary conditions $u(0,t)=u(\pi,t)=0$. Since the results we shall discuss do not depend on the direction of time, we assume w.l.o.g. that $\alpha>0$. We shall also assume that $\alpha\mu^2-\beta\mu+\gamma\neq 0$. By substituting $u=\exp(-\mu t)v$, we can transform to a problem with $\mu=0$.

We expand u in a Fourier series

$$u(x,t)=\sum_{n=1}^\infty a_n(t)\sin(nx), \tag{22}$$

and we obtain the ordinary differential equations

$$\alpha\ddot a_n+(\beta+n^2)\dot a_n+\gamma a_n=0. \tag{23}$$