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Nonsmooth feedback stabilizer for strict-feedback nonlinear systems that may not be linearizable at the origin

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Abstract

We present a continuous feedback stabilizer for nonlinear systems in the strict-feedback form, whose chained integrator part has the power of positive odd *rational* numbers. Since the power is not restricted to be larger than or equal to one, the linearization of the system at the origin may fail. Nevertheless, we show that the closed loop system is globally asymptotically stable (GAS) with the proposed continuous (but, possibly not differentiable) feedback. We formulate a condition that enables our design by characterizing the powers of the given system. The condition also shows that our result is an extension of Qian and Lin [Non-lipschitz continuous stabilizers for nonlinear systems with uncontrollable unstable linearization, Systems Control Lett. 42 (2001) 185–200] where the power of odd positive integers has been considered. New result on the global finite time stabilization problem is also presented.

Keywords: Power integrator; Finite time stabilization; Control Lyapunov function

1. Introduction

In practice, there exist systems that do not have the first approximation at the origin, e.g., a leaky bucket whose dynamics is given by $\dot{h} = -C\sqrt{h}$ [13, p. 41] or the hydraulic control systems [15]. Partly motivated by this fact, we construct a continuous (but possibly nondifferentiable) state feedback stabilizer which globally stabilizes a single-input nonlinear system in the strict-feedback form given by

$$\begin{aligned}\dot{x}_1 &= x_2^{r_1} + \phi_1(x_1)\\ \dot{x}_2 &= x_3^{r_2} + \phi_2(x_1, x_2)\\ &\vdots\\ \dot{x}_n &= u^{r_n} + \phi_n(x_1, \ldots, x_n),\end{aligned} \tag{1}$$

where $\phi_i(x_1, \ldots, x_i)$, $i = 1, \ldots, n$, are C^1 functions vanishing at the origin and r_i's are *rational numbers* whose numerators and denominators are all positive odd integers (we will call such r_i a positive odd rational number). We stress that if $r_i < 1$, then the system is not linearizable at each point $x \in \mathbb{R}^n$ with $x_{i+1}=0$ as well as at the origin, hence the standard backstepping design, which requires smoothness of the vector field, does not work directly.

This is a sharp contrast to the previous works [2–4,9,11,14] which have considered a system whose right-hand side is continuously differentiable in the state x, or all r_i's are greater than or equal to 1 so that its linearization at the origin may be uncontrollable. In [9,11,14], they propose a state feedback controller for the system (1) in which all r_i's are positive odd integers. Lin and Qian [9] explicitly construct, using a tool called adding a power integrator, a globally stabilizing *smooth* feedback control law for system (1) under the condition that the odd integer powers r_i are in decreasing order (i.e., $r_1 \geqslant \cdots \geqslant r_n \geqslant 1$), and under a growth condition that $|\phi_i(x_1, \ldots, x_i)| \leqslant (|x_1|^{r_i} + \cdots + |x_i|^{r_i})\gamma_i(x_1, \ldots, x_i)$, $i = 1, \ldots, n$, where each $\gamma_i(\cdot)$ is a smooth nonnegative function. The decreasing assumption and the growth condition have been removed in [4,11] while a continuous (instead of smooth) feedback is obtained in [11] and a smooth

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but time-varying feedback is designed in [4]. More generally, a triangular system

$$\dot{x}_i = f_i(x_1, \ldots, x_{i+1}), \quad i = 1, \ldots, n-1,$$

$$\dot{x}_n = f_n(x_1, \ldots, x_n) + u$$

is studied in [2,3], but it is assumed that all $f_i(\cdot)$'s are C^∞ so that its linearization at the origin does exist.

In this paper, we design a C^0 state feedback control law as well as a C^1 (positive definite and proper) Lyapunov function to make the origin globally asymptotically stable (GAS). The control law has an interesting feature: it contains some exponents (or powers) which are determined by a set of inequalities during the design procedure (thus, they are design parameters). These exponents are also closely related to the Lyapunov function used to prove the stability. The design procedure shows its efficiency when we discuss the finite time stabilization problem [1,5] since a slight change of the inequalities involved in the asymptotic stabilization problem ensures the existence of finite time stabilizer. One drawback of the proposed existence condition (the inequalities) is that it is not easy to check in general (although it can be converted to linear matrix inequality (LMI)). Thus, in order to avoid further difficulties, we provide explicit design guidelines for some special cases of r_i's.

The paper is organized as follows. In Section 2.1, we state our main theorem for global stabilization whose proof is given in Section 2.2. Global finite time stabilization problem is discussed in Section 2.3. In Section 2.4, the conditions proposed in the main theorems of Sections 2.1 and 2.3 are discussed in detail, where some relation to the previous work [11] is also pointed out. Finally, we conclude the paper in Section 3.

For convenience, let us define the set of all rational numbers whose numerators and denominators are all positive odd integers by Q_{odd}. Note that the set Q_{odd} is closed under multiplication, division and odd number of additions, but is not closed under even number of additions or subtraction, and that, therefore, x^{a+b} or $x^{c(a+b)}$ for $a, b, c \in Q_{\text{odd}}$ is a positive definite function of x.

2. Main results

2.1. Statement of main theorem

We now state our main theorem.

Theorem 1. *Suppose that, for the system* (1), $r_i \in Q_{\text{odd}}$, $i = 1, \ldots, n$. *If there exist* $\mu_0, \mu_1, \ldots, \mu_n \in Q_{\text{odd}}$ *such that*

$$\mu_0, \ldots, \mu_n \geqslant 1, \tag{2}$$

$$\frac{r_1}{\mu_1} \leqslant \frac{1}{\mu_0}, \quad \frac{r_2}{\mu_2} \leqslant \min\left\{\frac{1}{\mu_0}, \frac{1}{\mu_1}\right\}, \ldots, \quad \frac{r_n}{\mu_n} \leqslant \min\left\{\frac{1}{\mu_0}, \frac{1}{\mu_1}, \ldots, \frac{1}{\mu_{n-1}}\right\}, \tag{3}$$

$$0 \leqslant \frac{1}{\mu_0} - \frac{r_1}{\mu_1} \leqslant \frac{1}{\mu_1} - \frac{r_2}{\mu_2} \leqslant \cdots \leqslant \frac{1}{\mu_{n-1}} - \frac{r_n}{\mu_n}, \tag{4}$$

then there exists a C^0 *feedback controller* $u = u(x)$ *with* $u(0) = 0$ *which renders the origin of the closed loop system GAS. In addition, if the assumption holds with all* $\mu_i = 1$ $(i = 1, \ldots, n)$, *then a smooth feedback controller* $u(x)$ *exists.*

Remark 1. Note that from (4), the condition $\mu_i = 1$ $(i = 0, \ldots, n)$ implies $1 \geqslant r_1 \geqslant \cdots \geqslant r_n$. Note also that, once a set of μ_i's satisfying the conditions (2)–(4) is found, $(q\mu_i)$ with $q \in Q_{\text{odd}}$ also satisfies the conditions if (2) holds with them. On the other hand, the value of r_n does not restrict the existence a set of μ_i's for the conditions (because a sufficiently large μ_n can always be chosen), which is indeed related to the fact that the input term u^{r_n} of the system (1) can be simply replaced by another control v.

2.2. Constructive proof of the main theorem

In order to prove Theorem 1, we construct a feedback stabilizer through a modified backstepping procedure. We would like to point out that, unlike the conventional backstepping [8] or the construction of [11], the control Lyapunov functions at every step are chosen simultaneously considering the design of later steps to come. (See [4,7] for similar approaches.) To enable this, we have formulated, through the conditions of Theorem 1, a key property necessary for selecting a control Lyapunov function at each step. (Recall that the selection of μ_i is affected by the set of whole r_i's in the assumption.) In other words, we will use the values of μ_i, which have been obtained from (2)–(4), in the backstepping procedure.

Furthermore, we will frequently employ the following inequalities borrowed[1] from [11].

- For $x, y \in \mathbb{R}$ and $1 \leqslant q \in Q_{\text{odd}}$, we have

$$|x+y|^q \leqslant 2^{q-1}|x^q + y^q|. \tag{5}$$

- For $c, d, \rho \in \mathbb{R}$, if $c > 0$, $d > 0$ and $\rho > 0$, we obtain

$$|x|^c|y|^d \leqslant \frac{c}{c+d}\rho|x|^{c+d} + \frac{d}{c+d}\rho^{-\frac{c}{d}}|y|^{c+d}. \tag{6}$$

- For $a, b, c \in \mathbb{R}$, if $0 < a \leqslant b \leqslant c$, it is true that

$$|x|^b \leqslant |x|^a + |x|^c = |x|^a(1 + |x|^{c-a}), \quad x \in \mathbb{R}, \tag{7}$$

  because $|x|^b \leqslant |x|^a \leqslant |x|^a + |x|^c$ for $|x| \leqslant 1$ and $|x|^b \leqslant |x|^c \leqslant |x|^a + |x|^c$ for $|x| > 1$.
- Let $\phi_i : \mathbb{R}^i \to \mathbb{R}$ be a C^1 function with $\phi_i(0) = 0$. Then, there exists a smooth nonnegative function $\gamma_i(x_1, x_2, \ldots, x_i)$ such that

$$|\phi_i(x_1, \ldots, x_i)| \leqslant (|x_1| + \cdots + |x_i|)\gamma_i(x_1, \ldots, x_i). \tag{8}$$

[1] Inequalities (5) and (6) are proved (with slight extension) in a similar way to [11, Lemmas 2.3, 2.4], respectively, while inequality (8) is quite standard.