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Monotonicity properties for the viable control of discrete-time systems

Michel De Lara[a,*], Luc Doyen[b], Thérèse Guilbaud[c], Marie-Joëlle Rochet[c]

[a]CERMICS, Ecole des Ponts, ParisTech, 6-8 avenue Blaise Pascal, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France
[b]CNRS, CERESP (UMR 5173 CNRS-MNHN-P6), Muséum National d'Histoire Naturelle, 55 rue Buffon, 75005 Paris, France
[c]Département EMH (Ecologie et Modèles pour l'Halieutique), IFREMER, B.P. 21105, 44311 Nantes Cedex 03, France

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Abstract

This paper deals with the control of nonlinear systems in the presence of state and control constraints for discrete-time dynamics in finite-dimensional spaces. The viability kernel is known to play a basic role for the analysis of such problems and the design of viable control feedbacks. Unfortunately, this kernel may display very nonregular geometry and its computation is not an easy task in general. In the present paper, we show how monotonicity properties of both dynamics and constraints allow for relevant analytical upper and lower approximations of the viability kernel through weakly and strongly invariant sets. An example on fish harvesting management illustrates some of the assertions.

Keywords: Control; State constraints; Viability; Invariance; Monotonicity

1. Introduction

Let us consider a nonlinear control system described in discrete time by the difference equation

$$\begin{cases} x_{t+1} = f(x_t, u_t) & \forall\, t \in \mathbb{N}, \\ x_0 & \text{given}, \end{cases} \tag{1}$$

where the *state variable* x_t belongs to the finite-dimensional state space $\mathbb{X} = \mathbb{R}^{n_{\mathbb{X}}}$, the *control variable* u_t is an element of the *control set* $\mathbb{U} = \mathbb{R}^{n_{\mathbb{U}}}$ while the *dynamics* f maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$.

A controller or a decision maker describes "desirable configurations of the system" through a (non empty) set $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ termed the *desirable set*

$$(x_t, u_t) \in \mathbb{D} \quad \forall t \in \mathbb{N}, \tag{2}$$

where $\mathbb{D}$ includes both system states and controls constraints. Typical instances of such a desirable set are given by inequalities requirements: $\mathbb{D} = \{(x, u) \in \mathbb{X} \times \mathbb{U} | \forall i = 1, \ldots, p, \quad g_i(x, u) \geqslant 0\}$.

The *state constraints set* associated with $\mathbb{D}$ is obtained by projecting the desirable set $\mathbb{D}$ onto the state space $\mathbb{X}$:

$$\mathbb{V}^0 := \mathrm{Proj}_{\mathbb{X}}(\mathbb{D}) = \{x \in \mathbb{X} | \exists u \in \mathbb{U}, (x, u) \in \mathbb{D}\}. \tag{3}$$

Such problems of dynamic control under constraints refer to viability [1] or invariance [11] framework. Basically, such an approach focuses on inter-temporal feasible paths. It has been applied for instance to models related to the sustainable management of resource and bio-economic modeling as in [3–5,12,15,16,19]. From the mathematical viewpoint, most of viability and weak invariance results are addressed in the continuous time case. However, some mathematical works deal with the discrete-time case. This includes the study of numerical schemes for the approximation of the viability problems of the continuous dynamics as in [1,17]. Important contributions for the discrete-time case are also captured by the study of the positivity for linear systems as in [6], or by the hybrid control as in [2,20]. In the control theory literature, problems of constrained control lead to the study of positively invariant sets, particularly ellipsoidal and polyhedral ones for linear systems (see [9,13,14] and the survey paper [10]); reachability of target sets or tubes for nonlinear discrete-time dynamics is examined in [7].

Viability is defined as the ability to choose, at each time step $t \in \mathbb{N}$, a control $u_t \in \mathbb{U}$ such that the system configuration

* Corresponding author. Tel.: +33164153621; fax +33164153586.
E-mail address: delara@cermics.enpc.fr (M. De Lara).

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remains desirable. More precisely, the system is viable if the following set is not empty:

$$\mathbb{V}(f,\mathbb{D}) := \left\{ x_0 \in \mathbb{X} \;\middle|\; \begin{array}{l} \exists\ (u_0,u_1,\ldots) \text{ and } (x_0,x_1,\ldots) \\ \text{satisfying (1) and (2)} \end{array} \right\}. \quad (4)$$

The set $\mathbb{V}(f,\mathbb{D})$ is called the *viability kernel* associated with the dynamics f and the desirable set $\mathbb{D}$. By definition, we have $\mathbb{V}(f,\mathbb{D}) \subset \mathbb{V}^0 = \mathrm{Proj}_{\mathbb{X}}(\mathbb{D})$ but, in general, the inclusion is strict. For a decision maker or control designer, knowing the viability kernel has practical interest since it describes the states from which controls can be found that maintain the system in a desirable configuration forever. However, computing this kernel is not an easy task in general.

The present paper aims at giving explicit upper and lower approximations of this kernel using weakly (viable) or strongly invariant domains in the specific context of monotonicity properties of both constraints and dynamics.[1] To achieve this, let us recall what is meant by weakly or strongly invariant domains.

A subset $\mathbb{V}$ of the state space $\mathbb{X}$ is said to be *strongly invariant* for the dynamics f in the desirable set $\mathbb{D}$ if

$$\forall x \in \mathbb{V},\ \ \forall u \in \mathbb{U}, \quad (x,u) \in \mathbb{D} \Longrightarrow f(x,u) \in \mathbb{V}. \quad (5)$$

That is, if one starts from $\mathbb{V}$, any admissible control transfers the state into $\mathbb{V}$ providing, in any case, a desirable configuration. This is generally a too demanding requirement.

Similarly, a subset $\mathbb{V}$ is said to be *weakly invariant* for the dynamics f in the desirable set $\mathbb{D}$, or a *viability domain* of f in $\mathbb{D}$, if

$$\forall x \in \mathbb{V},\ \ \exists u \in \mathbb{U},\ \ (x,u) \in \mathbb{D} \quad \text{and} \quad f(x,u) \in \mathbb{V}. \quad (6)$$

That is, if one starts from $\mathbb{V}$, a suitable control may transfer the state in $\mathbb{V}$ and the system into a desirable configuration. In particular, it is worth pointing out that any desirable equilibrium is a viability domain of f in $\mathbb{D}$. A *desirable equilibrium* is an equilibrium of the system that belongs to $\mathbb{D}$, that is a pair $(\bar{x},\bar{u}) \in \mathbb{D}$ such that $\bar{x} = f(\bar{x},\bar{u})$.

Moreover, according to viability theory [1], the viability kernel $\mathbb{V}(f,\mathbb{D})$ turns out to be the union of all viability domains:

$$\mathbb{V}(f,\mathbb{D}) = \bigcup \left\{ \mathbb{V} \subset \mathbb{V}^0,\ \ \mathbb{V} \text{ viability domain for } f \text{ in } \mathbb{D} \right\}. \quad (7)$$

For the sake of completeness, we recall the proof in the Appendix (see Proposition 12). A major interest of such a property lies in the fact that any viability domain for the dynamics f in the desirable set $\mathbb{D}$ provides a *lower approximation* of the viability kernel.

An *upper approximation* $\mathbb{V}_k$ of the viability kernel is given by the so-called *viability kernel until time k associated with f in* $\mathbb{D}$:

$$\mathbb{V}_k := \left\{ x_0 \in \mathbb{X} \;\middle|\; \begin{array}{l} \exists\ (u_0,u_1,\ldots,u_k) \text{ and } (x_0,x_1,\ldots,x_k) \\ \text{satisfying (1) for } t=0,\ldots,k-1 \\ \text{and (2) for } t=0,\ldots,k \end{array} \right\}. \quad (8)$$

[1] No topological assumptions are needed. Only for Proposition 8, we do require a continuity property.

We have

$$\mathbb{V}(f,\mathbb{D}) \subset \mathbb{V}_{k+1} \subset \mathbb{V}_k \subset \mathbb{V}_0 = \mathbb{V}^0 \quad \forall k \in \mathbb{N}. \quad (9)$$

It may be seen by induction that the decreasing sequence of viability kernels until time k satisfies the following dynamic programming equation:

$$\mathbb{V}_0 = \mathbb{V}^0 \quad \text{and} \quad \mathbb{V}_{k+1} = \{x \in \mathbb{V}_k \mid \exists u \in \mathbb{U},\ f(x,u) \in \mathbb{V}_k \text{ and } (x,u) \in \mathbb{D}\}. \quad (10)$$

By (9), such an algorithm provides approximation from above of the viability kernel as follows:

$$\mathbb{V}(f,\mathbb{D}) \subset \bigcap_{k \in \mathbb{N}} \mathbb{V}_k = \lim_{k \to +\infty} \downarrow \mathbb{V}_k. \quad (11)$$

In [1], conditions for the equality to hold true are exposed (are required the compacity for the constraints and upper semicontinuity with closed images for the set-valued map associated with the controlled dynamics).

Once the viability kernel, or any approximation, or a viability domain is known, we have to consider the management or control issue, that is the problem of selecting suitable controls at each time step. For any viability domain $\mathbb{V}$ and any state $x \in \mathbb{V}$, the following subset $\mathbb{U}_{\mathbb{V}}(x)$ of the decision set $\mathbb{U}$ is not empty:

$$\mathbb{U}_{\mathbb{V}}(x) := \{u \in \mathbb{U} | (x,u) \in \mathbb{D} \text{ and } f(x,u) \in \mathbb{V}\}. \quad (12)$$

Therefore, $\mathbb{U}_{\mathbb{V}(f,\mathbb{D})}(x)$ stands for the largest set of *viable controls associated with* $x \in \mathbb{X}$. Then, the decision design consists in the choice of a viable *feedback* control, namely any selection $\Psi : \mathbb{X} \to \mathbb{U}$ which associates with each state $x \in \mathbb{V}(f,\mathbb{D})$ a control $u = \Psi(x)$ satisfying $\Psi(x) \in \mathbb{U}_{\mathbb{V}(f,\mathbb{D})}(x)$.

The paper is organized as follows. Section 2 is devoted to the definitions of monotonicity for both the dynamics and constraints. Then, Section 3 exhibits lower and upper approximations of the viability kernel in this monotonicity context. An example is exposed in Section 4 to illustrate some of the main findings.

2. Monotonicity properties

In this section we define what is meant by monotonicity of the desirable set $\mathbb{D}$ together with the dynamics f, both with respect to state x and control u.

2.1. Set monotonicity

In what follows, the state space $\mathbb{X}$ and the control space $\mathbb{U}$ are $\mathbb{X} \subset \mathbb{R}^{n_{\mathbb{X}}}$ and $\mathbb{U} \subset \mathbb{R}^{n_{\mathbb{U}}}$ supplied with the componentwise order: $x' \geqslant x$ if and only if each component of x' is greater than or equal to the corresponding component of x:

$$x' \geqslant x \iff x'_i \geqslant x_i,\ \ i = 1,\ldots,n.$$

We also define the maximum $x \vee x'$ of (x,x') as follows:

$$x \vee x' := (x_1 \vee x'_1,\ldots,x_n \vee x'_n) = (\max(x_1,x'_1),\ldots,\max(x_n,x'_n)).$$