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Well-defined steady-state response does not imply CICS

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Abstract

Systems for which each constant input gives rise to a unique globally attracting equilibrium are considered. A counterexample is provided to show that inputs which are only asymptotically constant may not result in states converging to equilibria (failure of the converging-input converging state, or "CICS" property).

Keywords: Step response; Global attractivity; Homoclinic orbits; Asymptotically autonomous systems

1. Introduction

Consider a controlled finite dimensional system

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

under suitable regularity assumptions, and assume that the following property holds: for each constant input $u \equiv a$, there is a unique steady-state x_a (that is, $f(x, a) = 0$ has the unique solution $x = x_a$), and every solution of the system $\dot{x} = f(x, a)$ converges to this state x_a. In the terminology of [1–3], we say that the system has a "characteristic" or a *monostable steady-state step response*.

Given that (1) admits a characteristic, it is natural to ask if the following *converging-input converging-state* (*CICS*) property must then also hold: for every convergent input $u(\cdot)$ (that is to say, $u(t) \to a$ as $t \to \infty$, for some value a), every bounded solution of $\dot{x} = f(x, u)$ converges to x_a.

Such a property is especially interesting when studying cascades of systems, in which the input u to the system being studied is itself the output of another system. That is, there is another system $\dot{z} = g(z, v)$, $u = h(z)$, and v is an external input to the cascade. In that context, one would like to know whether each of the f and g systems having a characteristic implies that the cascade also does. Suppose that $v \equiv a$. If the g system has a monostable response, its state converges to some value: $z(t) \to z_a$, so that also, assuming continuity of the read-out map h, $u(t) \to b := h(z_a)$. If the CICS property holds for the f subsystem (and assuming that its trajectories are bounded), then $x(t) \to x_b$, and therefore the complete state $(z(t), x(t))$ converges to (z_a, x_b), establishing that the cascade also admits a characteristic.

These questions have a long history in control as well as in dynamical systems theory, see for example the early work of Markus [6], and are closely related to the topic of "asymptotically autonomous" systems, see for example [4] Appendix A (by Z. Artstein). The latter are time-varying systems $\dot{x} = F(x, t)$ for which $F(x, t) \to F_0(x)$ as $t \to \infty$, for some time-invariant vector field F_0, where the convergence is assumed to hold in an appropriate technical sense. Clearly, one may view $f(x, u(t))$, for any fixed given input $u(\cdot)$, as a time-varying vector field $F(x, t)$, and, if $u(t) \to a$ as $t \to \infty$, one may define $F_0(x) := f(x, a)$; in this manner, "$u(t) \to a$" translates into "$F(x, t) \to F_0(x)$," and the questions addressed here amount to relating the behaviour of solutions of $\dot{x} = F(x, t)$ to that of solutions of the limit system $\dot{x} = F_0(x)$. For other related work, see for example [7–9,11,12,5].

There are several known sufficient conditions that guarantee the CICS property for systems which admit characteristics. One such condition is *stability* of the equilibria x_a. That is, not only

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do trajectories of $\dot{x} = f(x, a)$ approach x_a as $t \to \infty$, but the "small excursion" Lyapunov stability condition holds as well: for each neighbourhood U of x_a, there is another neighbourhood V such that solutions starting in V do not exit U (later in this paper we discuss a weaker condition, which is implied by but does not imply the stability condition). The conjunction of stability and global attractivity of x_a is, of course, equivalent to global asymptotic stability of x_a under which condition the CICS property is a particular consequence of Theorem 2 in [6]. A different condition ensuring the CICS property is that of *monotonicity*: the conclusions hold provided that the system is monotone as an input/output system in the sense of [2]; the paper [2] made stability into part of the definition of characteristic, but [3] showed one need not assume stability in order to conclude the CICS property.

In view of these different sufficient conditions, it is natural to ask if it is always true that the CICS property holds for systems with characteristics. The main goal of this note is to provide a negative answer to that question by means of a counterexample. The counterexample is two-dimensional (one-dimensional systems are always monotone, so no one-dimensional counterexamples could exist) and quite explicit. The construction is provided in the next section. For completeness, in the last section we review a simple criterion which guarantees the CICS property for systems with characteristics.

2. The counterexample

It is well known that large-time behaviour of solutions of an asymptotically autonomous system can differ markedly from that of solutions of its autonomous limit system: examples can be found in [11,12]. In the context of the present note, the property that, for every constant input, there should exist a unique attracting state is a distinguishing feature that adds subtlety to the planar counterexample, the construction of which can be summarised as follows.

First, we determine a locally Lipschitz function $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ such that, for every constant $a \in \mathbb{R}$, (0, 1) is a globally attractive equilibrium of the autonomous system $\dot{x} = f(x, a)$. For each $r^0 > 1$, we then proceed to construct an input u, with $u(t) \to 0$ as $t \to \infty$, such that, for all initial data $x^0 \in \mathbb{R}^2$ with Euclidean norm $|x^0| = r^0$, the solution of the initial-value problem $\dot{x} = f(x, u)$, $x(0) = x^0$, is bounded and has the unit circle S^1 as its ω-limit set.

In order to define our system, we start by introducing an auxiliary function. Let $h : \mathbb{R} \to [0, \infty)$ be any smooth function such that $h(y) = 1$ for all $y \in [1, 2]$ and $h(y) = 0$ for all $y \notin [\frac{1}{2}, 4]$, and let $g : [0, \infty) \times \mathbb{R} \to [0, \infty)$ be given by

$$g(r, u) := \begin{cases} u^p h((r-1)/u), & u > 0, \\ 0, & u \leqslant 0, \end{cases} \tag{2}$$

with constant $p > 2$. We remark that g is a locally Lipschitz function. To see this, note that g is smooth on each of $[0, \infty) \times (0, \infty)$ and $[0, \infty) \times (-\infty, 0)$, and is continuous on all of $[0, \infty) \times \mathbb{R}$. Let k be an upper bound on h and $|h'|$ (such a bound exists since $h(y) = 0 = h'(y)$ for all $y \notin [\frac{1}{2}, 4]$). Then, for all (r, u) with $0 < |u| \leqslant 1$, we have $|\partial g/\partial r(r, u)| \leqslant k$, $|\partial g/\partial u(r, u)| \leqslant (p+4)k$, and $|\nabla g(r, u)| \leqslant k\sqrt{1 + (p+4)^2}$, which implies that g is uniformly Lipschitz around $u = 0$.

Consider the system on $\mathbb{R}^2$ (with Euclidean norm $|\cdot|$)

$$\dot{x} = f(x, u),$$

with $f = (f_1, f_2) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ given by

$$f_1(x, u) := \begin{cases} -x_1(|x|-1)^{p+1}/|x| \\ \quad -2x_2(1 - (x_1/|x|)) \\ \quad -x_2 g(|x|, u), & |x| \geqslant 1, \\ 2(x_1 - 1)x_2, & |x| < 1, \end{cases}$$

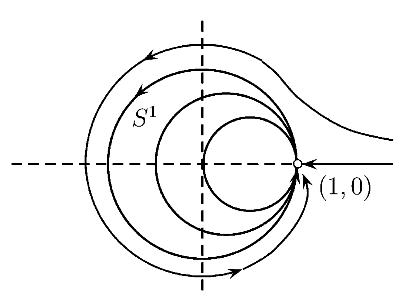
$$f_2(x, u) := \begin{cases} -x_2(|x|-1)^{p+1}/|x| \\ \quad +2x_1(1 - (x_1/|x|)) \\ \quad +x_1 g(|x|, u), & |x| \geqslant 1, \\ -(x_1 - 1)^2 + x_2^2, & |x| < 1 \end{cases}$$

(where $x = (x_1, x_2)$). We claim that f is locally Lipschitz. Since g is locally Lipschitz, to prove the claim it suffices to show that the u-independent vector field F given by $F(x) := f(x, u) - (-x_2, x_1)g(|x|, u)$ is locally Lipschitz. Let S^1 denote the unit circle centred at 0 in $\mathbb{R}^2$. Observe that F is continuous at all points $(x, u) \in S^1 \times \mathbb{R}$ and continuously differentiable on $(\mathbb{R}^2 \backslash S^1) \times \mathbb{R}$ with bounded derivative on $(K_1 \backslash S^1) \times K_2$ for every compact neighbourhood K_1 of S^1 and every compact $K_2 \subset \mathbb{R}$. It follows that F is locally Lipschitz.

With zero input $u = 0$ ($a = 0$ in the notation in the Introduction), the system has a globally attractive (but not stable) equilibrium at $x_a = (1, 0)$, as will follow from a more general result shown below for arbitrary constant inputs. In particular, $S^1 \backslash \{(1, 0)\}$ is a homoclinic connection of (1, 0) to itself. More generally, the collection of punctured circles

$$\{x \mid |x - (b, 0)| = 1 - b\} \backslash \{(1, 0)\}, \quad 0 \leqslant b < 1,$$

constitutes a family of such homoclinic connections filling the unit disc. For all inputs u, this family of homoclinic connections persists (as the vector field on the closed unit disc coincides with the zero input case).



Exterior to the open unit disc, the system representation, in polar coordinates, is given by

$$\dot{r} = -(r-1)^{p+1}, \quad \dot{\theta} = 4\sin^2(\theta/2) + g(r, u). \tag{3}$$

Therefore, in view of (2), for every *constant* input $u = a \neq 0$ the vector field differs from the zero-input case only when $a > 0$ and