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A high-resolution shallow water model using unstructured quadrilateral grids

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ABSTRACT

A two-dimensional cell-centred finite volume model for quadrilateral grids is presented. The solution methodology of the depth-averaged shallow water equations is based upon a Godunov-type upwind finite volume formulation, whereby the inviscid fluxes of the system of equations are obtained using the HLL Riemann solver. A simple yet precise analytical expression is presented to compute hydrostatic flux through an interface of a quadrilateral cell in order to achieve exact balance between flux gradient and bed slope source terms under still water condition. A multidimensional gradient reconstruction procedure and a continuously differentiable multidimensional slope limiter based on a wide computational stencil are proposed to maintain second-order spatial accuracy. The proposed second-order scheme is shown to be more accurate even when distorted grids are used and is therefore more suitable for practical applications. The presented model is verified and validated by solving a wide variety of test cases having analytical solutions and laboratory measurements.

1. Introduction

The two-dimensional (2D) shallow water equations are numerically solved to simulate many hydraulic and environmental engineering flow problems. A great amount of literature exists describing various computational techniques such as the finite difference method [1,2], the finite element method [3,4] and the finite volume method [5–7], which have been developed to obtain satisfactory solutions of the system of equations. The use of the finite volume technique has become more popular in recent times for simulating free surface flows because of its simplicity of implementation and good flexibility for space discretization. In addition, the finite volume method is based on the integral form of the conservation equations, and thus a scheme in conservation form can easily be constructed to capture shock waves (shock-capturing property).

Godunov-type finite volume solvers of the shallow water equations have a shock-capturing property that is essential to preserve discontinuous or steeply varying gradients that occur in transcritical and sharp-fronted shallow flows. Typical examples of Godunov-type flow models can be found in literature [e.g. 5–11]. By upwinding the flux within the integral conservation form of the governing equations, Godunov-type methods represent physically correct propagation of information throughout the flow field by solving sets of Riemann problems over the entire flow domain. Many Riemann solvers now exist, out of which the HLL solver is preferred here because it better describes the flux for dry bed situation and does not require any entropy fix [8,12]. Although, the HLL Riemann solver is robust, difficulties arise in solving the Riemann problem when source terms are included in the governing equations. Essentially, a numerical imbalance is created by artificially splitting the surface gradient into flux gradient and bed slope terms. Improper discretization of these terms does not result in a well-balanced scheme. Bermúdez and Vázquez-Cendón [13] proposed upwinding of the source terms to achieve equilibrium between flux gradients and source terms, and this method significantly improved the accuracy of the numerical solution, compared with earlier methods. Later, Vazquez-Cendon [14] modified the same idea to solve more general flow problems in the case of a one-dimensional channel with longitudinal width variations. LeVeque [15] proposed a well-balanced scheme by introducing a Riemann problem inside a cell to account for the propagation of source terms, but this has been reported to be less robust when predicting steady transcritical flows that contain shocks. Burguete and Garcia-Navarro [16] proposed conservative schemes with flux adjusted source term discretization technique using either a semi-implicit or upwinding method. Zhou et al. [10] developed the Surface Gradient Method (SGM) and pointed out that the main source of error is caused by inaccurate reconstruction of water depth. Valiani and Begnudelli [17] presented a novel method to compute the bed slope source terms from the pressure terms evaluated for all the faces of an n-sided cell. Another simple balancing approach was presented by Kuiry et al. [11,18] for triangular and Cartesian grids by introducing an improved treatment of pressure

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term. However, most of the literature on well-balanced schemes are limited to Cartesian and triangular grids. The development of well-balanced scheme for unstructured quadrilateral grids is relatively difficult and hence the simple analytical method of Kuiry et al. [11,18] has been modified and presented herein.

The performance of Riemann based approaches greatly depends on temporal and spatial accuracy. The use of piecewise constant data for the first-order accurate schemes often does not fulfil the desired accuracy. Therefore, higher-order schemes are developed by reconstructing the conservative variables within a cell. The reconstruction procedure involves computations of gradients of the variables based on the cell-centred values in a defined stencil. While generating grids using a mesh generator distorted grids in some regions may be automatically generated. Nevertheless, a grid can be further distorted due to the presence of steep gradients in the bottom topography. The MUSCL based one-dimensional reconstruction approach [19] has been quite effective for structured grids but may produce poor results on distorted grids [20–22]. An alternative approach could be the use of multidimensional reconstruction method based on the cell-centred and cell-vertex formulations on unstructured grids. Such a reconstruction method possesses dependence on a wide computational stencil and does not strongly depend on vertex values to preserve stability even for distorted grids. However, the higher-order schemes often produce nonphysical oscillations. These oscillations can be suppressed by limiting the slopes of the reconstructed variables using a nonlinear function called a limiter. For unstructured grids, the limiter should be inherently multidimensional and a one-dimensional approach may not be suitable. Jawahar and Kamath [21] and Yoon and Kang [22] proposed multi-dimensional reconstruction procedure and multidimensional limiter for triangular grids but similar developments for quadrilateral grids are not found in literature. In the present study, a gradient reconstruction method and a multidimensional limiter suitable for regular and distorted quadrilateral grids are presented. The proposed second-order method is shown to be producing better results when distorted grids are used.

The study presents a cell-centred finite volume model on quadrilateral grids. In order to achieve exact balance between flux gradients and source terms, a simple algebraic method is proposed. For higher-order accuracy, a multidimensional reconstruction technique and a continuously differentiable multidimensional limiter are introduced for unstructured quadrilateral grids. The model is applied to a number of analytical and experimental test problems and the computed results are investigated for comparative study.

2. Governing equations

The two-dimensional, shallow water mathematical model is obtained by integrating the Navier–Stokes equations over the flow depth. The assumptions made in the process are: incompressible fluid, uniform velocity distribution in the vertical direction, hydrostatic pressure distribution and small bottom slope.

Neglecting diffusion of momentum due to viscosity and turbulence, wind effects and the Coriolis term, the continuity and momentum equations in conservation form can be expressed as [23]:

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}+\frac{\partial \mathbf{G}(\mathbf{U})}{\partial y}=\mathbf{S}(x,y,\mathbf{U}) \tag{1}$$

in which

$$\mathbf{U}=(H,q_x,q_y)^T$$

$$\mathbf{F}=\left(q_x,\frac{q_x^2}{h}+\frac{gh^2}{2},\frac{q_xq_y}{h}\right)^T$$

$$\mathbf{G}=\left(q_y,\frac{q_xq_y}{h},\frac{q_y^2}{h}+\frac{gh^2}{2}\right)^T$$

$$\mathbf{S}=\left(0,gh(S_{0x}-S_{fx}),gh(S_{0y}-S_{fy})\right)^T$$

where $\mathbf{U}$ represents the vector of conserved variables, $\mathbf{F}$ and $\mathbf{G}$ are the fluxes associated with the conserved variables in the x- and y-directions. In addition, $q_x = uh$ and $q_y = vh$ are the unitary water discharges, u and v are depth-averaged velocities in the x- and y-directions, respectively, while $S_{ox} = -\partial z_b/\partial x$ and $S_{oy} = -\partial z_b/\partial y$ are the bed slopes along these directions. The variable $H = h + z_b$ is the water level, h represents depth of flow, z_b defines bottom elevation, g is the acceleration due to gravity. Here, water level (H) is considered as one of the unknown variables instead of water depth (h) due to the fact that reconstruction of h for higher-order spatial accuracy directly from the cell average values will not guarantee a continuous reconstruction at the cell boundaries if the bed slope varies from cell to cell [10,24]. The friction slopes are estimated using the Manning relation [1,5,25] as given below.

$$S_{fx}=n^2u\sqrt{u^2+v^2}h^{-4/3};\quad S_{fy}=n^2v\sqrt{u^2+v^2}h^{-4/3} \tag{2}$$

where n is the Manning's roughness coefficient. In general, the influence of bottom roughness prevails over the turbulent shear stress between cells. Therefore the effective stress terms are neglected in the computations.

3. Numerical solution

The computational domain is divided into a finite number of unstructured quadrilateral cells which form the control volumes. Eq. (1) is then integrated over an elementary control volume and discretized by finite volume method. The dependent variables of the system are assumed to be stored at the centre of the cell and represented as piecewise constants. It is useful to rewrite Eq. (1) as

$$\frac{\partial \mathbf{U}}{\partial t}+\nabla\cdot\mathbf{E}(\mathbf{U})=\mathbf{S}(x,y,\mathbf{U}) \tag{3}$$

where $\mathbf{E} = (\mathbf{F}, \mathbf{G})^T$, the flux tensor, is introduced in order to express the integral form of the equations over a fixed volume V,

$$\frac{\partial}{\partial t}\int_V \mathbf{U}dV+\int_V(\nabla\cdot\mathbf{E})dV=\int_V \mathbf{S}dV \tag{4}$$

For the computational domain defined by a set of quadrilateral cells, a discrete approximation to Eq. (4) is applied over each cell V so that the volume integrals represent integrals over the cell with the dependent variables represented as piecewise constants. The Gauss divergence theorem is applied to the second integral of Eq. (4) and the contour integral is approached via a mid-point rule, that is, a numerical flux is defined at the mid-point of each edge giving

$$\oint_\xi(\mathbf{F}n_x+\mathbf{G}n_y)d\xi=\sum_{e=1}^{4}\mathbf{E}_e^*\cdot\mathbf{n}_e d\xi_e \tag{5}$$

where e is the cell edge index and $\mathbf{E}^*$ is the numerical flux vector, ξ is the boundary of the plan area A of the control volume V, n_x and n_y are the components of unit normal in the x- and y-directions, respectively. The explicit expression of $\mathbf{E}^*$ depends upon the selected *Riemann solver* [5,7,12,26–31]. In the present work, the HLL Riemann solver of Harten, Lax, and van Leer [12,28] is used to compute the flux. It is preferred to Roe's Riemann solver because it better describes the flux for dry bed situation and does not require any entropy fix [8,12].

The HLL scheme assumes one constant state between left and right waves and defines the flux at an interface as