Computers & Fluids 68 (2012) 219–243

Contents lists available at SciVerse ScienceDirect

Computers & Fluids

journal homepage: www.elsevier.com/locate/compfluid

An auto-adaptive approximate Riemann solver for non-linear Euler equations

Capdeville Guy *

Département de Mécanique des Fluides, Ecole Centrale de Nantes, 1, Rue de la Noë, B.P. 92101, 44321 Nantes Cedex 3, France

A R T I C L E   I N F O

Article history:
Received 1 February 2012
Received in revised form 16 May 2012
Accepted 3 July 2012
Available online 16 July 2012

Keywords:
Hyperbolicity
Least-square
MUSCL
Rankine-Hugoniot
WENO
Finite-volume
Gas dynamics
HLL
Upwind differencing
Entropy inequalities
Positive conservation
Riemann solver

A B S T R A C T

We present a new HLL-type approximate Riemann solver that aims at capturing any isolated discontinuity without necessitating extensive characteristic analysis of governing partial differential equations. This property is especially attractive for complex hyperbolic systems with more than two equations. Following Linde's approach [6], we introduce a generic middle wave into the classical two-state HLL solver. The property of this third wave is typified by the way of a "strength indicator" that is derived from polynomial considerations. The polynomial that constitutes the basis of the procedure is made non-oscillatory by an adapted fourth-order WENO algorithm (CWENO4). This algorithm makes it possible to derive an expression for the strength indicator. According to the size of this latter parameter, the resulting solver (HLL-RH), either computes the multi-dimensional Rankine-Hugoniot equations if an isolated discontinuity appears in the Riemann fan, or asymptotically tends towards the two-state HLL solver if the solution is locally smooth. The asymptotic version of the HLL-RH solver is demonstrated to be positively conservative and entropy satisfying in its first-order multi-dimensional form provided that a relevant and not too restrictive CFL condition is considered; specific limitations of the conservative increments of the numerical solution and a suited entropy condition enable to maintain these properties in its high-order version.

With a monotonicity-preserving algorithm for the time integration, the numerical method so generated, is third order in time and fourth-order accurate in space for the smooth part of the solution; moreover, the scheme is stable and accurate when capturing a shock wave, whatever the complexity of the underlying differential system.

Extensive numerical tests for the one- and two-dimensional Euler equation of gas dynamics and comparisons with classical Godunov-type methods help to point out the potentialities and insufficiencies of the method.

1. Introduction

Computing numerical solutions of hyperbolic systems of conservation laws is a challenging work. Because non-linear hyperbolic partial differential equations may give rise both to discontinuous solutions as shock waves or contact surfaces and sonic expansion waves, any advanced numerical method must be able to capture these features without introducing spurious oscillations or losing the physical meaning of the solution.

Over the past three decades, Godunov's method [3] and its numerous derivatives [29], have been employed successfully as numerical solvers for non-linear hyperbolic problems. The great popularity of these schemes is primarily due to their robustness, the possibility of achieving high resolution of stationary discontinuities and the availability of an underlying physical model.

Central to these methods is the solution of the Riemann problem that naturally arises in the conservative discretization of advection problems. However, exact solution of the Riemann problem requires the use of an iterative procedure that leads to relatively complex and time-consuming numerical schemes. To overcome this drawback, several approximations to the Riemann problem have been devised [4,29]. Thus, many of those resulting upwind schemes are able to produce nice results on specific problems; however, certain approximations can also fail dramatically or produce numerical side effects [5,33].

Among these methods, the approximate solver–initially suggested by Harten, Lax and Van-Leer (HLL) in 1983 [2], forms the basis of very efficient and robust approximate Godunov-type methods. With this approach, the main idea is to assume for the solution, a wave configuration that consists of two extreme waves separating three constant states, whatever the complexity of the physics. With an appropriate choice of wave velocities [4], the simplest HLL approximation resolves isolated shock waves [4,10], is positively conservative [5] and satisfies an entropy inequality [2].

Unfortunately, the assumption that is the basis of the HLL approach (two-wave model) is correct only for hyperbolic systems of two equations such as, for example, the one-dimensional shallow water equations. In gas dynamics or magneto-hydrodynamics, this two-wave assumption becomes incorrect. Consequently, the

* Tel.: +33 2 4037 1651; fax: +33 2 4037 2523.
E-mail address: guy.capdeville@ec-nantes.fr

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http://dx.doi.org/10.1016/j.compfluid.2012.07.001

resolution of all physical intermediate waves between the two extreme waves becomes inaccurate; for the limiting case for which these intermediate waves are stationary relative to the mesh, the resulting numerical smearing becomes even excessive [29]. A breakthrough in this domain came from Toro, Spruce and Speares [11], who proposed the so-called "HLLC Riemann solver" ("C" standing for "contact") as applied to the one-dimensional time-dependent Euler equation of gas dynamics. This method assumes a three-wave model, resulting in better resolution of the intermediate wave (the contact discontinuity for the one-dimensional Euler equations). Later on, Batten et al. [10], made a thorough analysis of the HLLC scheme and suggested new ways of estimating the characteristic wave speeds; they also showed that the resulting numerical method is positively conservative, resolves isolated shock and contact waves, exactly, when discretizing the one-dimensional Euler equations. Since Euler equations have three distinct characteristic fields, whatever the spatial dimension considered, the HLLC approach is a complete Riemann solver, in that case.

However, for systems with eigenstructure containing more than three distinct characteristic fields, the HLLC method becomes incomplete and behaves similarly to HLL for the one-dimensional Euler equations.

Currently, there exists two ways for extending the HLLC approach:

The first one is to admit the correct number of characteristic fields for the system of interest by increasing the number of intermediate waves. This is, for example, the line followed in [7]or [8,9]. Another solution due to Linde [6], is to design an HLL-type method that is not tied to specific characteristic properties of the governing equations. This can be done, for example, by introducing into the initially two-state HLL Riemann solver, a generic middle wave of which the strength is sized by a heuristic normalized parameter. On one hand, if the solution is locally smooth, this parameter is 0 and the resulting numerical flux returns to the single-state HLL flux. On the other side, if an isolated discontinuity exists into the Riemann fan, this parameter becomes 1 and the middle wave connects the left and right states of the initial Riemann problem, according to the theory developed in [2].

In [6], the "strength indicator" of the middle wave is obtained through the least-square solution of the one-dimensional Rankine-Hugoniot jump condition with appropriate re-scaling; in [12], another strategy is devised for computing this parameter.

With Linde's approach, the resulting solver makes it possible to resolve all physically admissible isolated discontinuities and can be applied for a complex system without necessitating a detailed knowledge of its characteristic field.

The work we present in this article follows this latter line of thought and tries to cure some of its deficiencies.

Indeed, Linde's flux may generate unphysical oscillations due to the specific definition of the strength indicator of the middle wave; this is the reason why in [12], a reformulation of this parameter was designed to smooth unwanted oscillations while preserving the positivity of density and internal energy. The resulting solver is a positively conservative variant of Linde's solver. In both works [6,12], the strength indicator is intrinsically one-dimensional since it was defined from the left and right states of the solution at the cell interfaces.

In this article, the solution we propose for defining this crucial parameter is based upon polynomial considerations and is suited for multi-dimensional computations. For this purpose, we utilize an extension of the multi-dimensional CWENO interpolation procedure, devised in [18] from the ideas of [13,14,16,17], and initially designed to produce a high-order monotonicity-preserving MUSCL-like scheme.

Indeed, to avoid Gibbs-like phenomena in presence of discontinuities, this interpolation procedure uses an adaptive stencil based upon a "smoothness indicator" of the solution [13,14]. Thus, by a suited re-normalization, we define this smoothness indicator as the strength indicator of the middle wave. By doing so, when there is a discontinuity, this indicator becomes close to 1 and the scheme solves the multi-dimensional Rankine-Hugoniot equations in a least-square sense, this in the normal directions to the mesh. Alternatively, when the solution is smooth, the strength indicator asymptotically tends towards 0 and, therefore, the contribution due to the middle wave becomes negligible: the resulting solver resembles the classical HLL-scheme.

Theoretically, the advantage of such a choice is twofold: firstly, in contrast with above mentioned solutions, the strength indicator is now defined in a multi-dimensional way; secondly, its definition is closely tied with the non-oscillatory properties of the interpolation polynomial used in the MUSCL procedure of the scheme. This way, we can design a HLL-type Riemann solver of which the numerical characteristics continuously and dynamically adapt to the regularity of the solution. This behavior is expected to smooth out the spurious oscillations that might be generated using Linde's approach, while preserving its accuracy properties in presence of isolated discontinuities.

Finally, to characterize the resulting scheme completely, it remains to check its stability properties. Indeed, the difficulty for high-order schemes to handle complex flow conditions, necessitates adding one more criterion to the design of an approximate Riemann solver: the positive conservation. Initially introduced by Einfeldt et al. [5] the term "positively conservative" refers to a conservative scheme that predicts positive density and pressure for all time if data considered are physically meaningful. The positivity of a numerical scheme is a weak stability principle that does not prevent some forms of instability such as a Gibbs-like phenomenon but makes it possible to derive the maximum allowable time-step. In addition, it is significant that such a stability property is supplemented by a set of discrete entropy inequalities in order to discard non-physical solutions. In [19] Perthame-Shu provided a general framework and illustrated the way to impose positivity of density and pressure for finite volume schemes, for one and two space dimensions and for first and third order accuracy, by starting from a positivity-preserving approximate 1D Riemann solver. Later on, following these ideas, some variants were proposed [20–23] in order to generalize this work. In addition, second-order entropy inequalities were demonstrated for some second-order 1D [22] or 2D [21] schemes as long as a relevant CFL condition and a limitation procedure are considered.

In this paper, starting from the stability properties of the asymptotic form for the 1D version of the first-order HLL-RH scheme, we establish that these properties are sufficient to guarantee positive conservation for the resulting two-dimensional, first-order, finite-volume scheme, on arbitrary grids. We demonstrate that the time-step limitation to reach this result is not too restrictive compared to the initial 1D condition. In addition, we demonstrate that the scheme then verifies a set of entropy inequalities if a simple entropy condition is added for the one-dimensional version of the scheme. Extension of these properties to high-order accuracy is promoted by applying a limitation procedure to the high-order conservative increments of the numerical solution. To this aim, the limitation procedure initially devised in [19] and generalized in [21,22], is adapted to our scheme.

The resulting numerical method applies to the compressible Euler equations.

The outline of this article is as follows: in Section 2, some fundamentals and notations concerning the finite volume framework of the method, are given. In Section 3, the procedure to generate the new approximate Riemann solver (HLL-RH) is described; the least-square method that makes it possible to solve the multi-dimensional Rankine-Hugoniot equations, is detailed. Section 4 is devoted