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Optimum blade loading for a powered rotor in descent

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Abstract The optimum loading for rotors has previously been found for hover, climb and wind turbine conditions; but, up to now, no one has determined the optimum rotor loading in descent. This could be an important design consideration for rotary-wing parachutes and low-speed descents. In this paper, the optimal loading for a powered rotor in descent is found from momentum theory based on a variational principle. This loading is compared with the optimal loading for a rotor in hover or climb and with the Betz rotor loading (which is optimum for a lightly-loaded rotor). Wake contraction for each of the various loadings is also presented.

1. Introduction

A problem of long-standing interest in rotor and propeller theory has been determination of the optimum blade loading for a rotor (i.e., the loading that gives minimum power for a given thrust). Glauert's second approximation to momentum theory allows him to invoke a variational principal to obtain the optimality condition for a rotor in hover.[1] He also works out a numerical approximation.[1] Glauert's minimum power is demonstrated in Ref. [1] to be slightly lower than the power due to the Betz loading.[2] Ref. [3] demonstrates that Glauert's variational principle for hover can be cast as a cubic equation in the unknown loading that has a compact, closed-form solution for the optimum blade loading in hover (as based on momentum theory).

Ref. [4] offers a third approximation to Glauert's momentum equations. This third approximation gives the same optimality condition as does Glauert's second approximation, but it allows development of wake contraction equations—valid in hover and climb—to give downstream variables due to an arbitrary loading distribution. Applications are given in Ref. [4] for the Betz loading distribution. Ref. [4] also demonstrates that, for powered rotors in descent, the Betz loading always results in some portions of the rotor being in either wind-turbine state or vortex-ring state. Thus, solution of the contraction equations with a Betz distribution is not possible for descent. It is further found in Ref. [4] that—beyond a critical descent rate—no portion of a Betz-loaded rotor is in a working state (i.e., momentum theory breaks down over the entire span, and one enters the vortex-ring region from the helicopter side). This critical descent rate with the Betz distribution is shown to be the same descent rate predicted by the vortex theory of Wolkovitch.[5]

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Ref. [6] reveals that Glauert's variational principal for optimum loading can be extended to the case of general climb or descent rate and leads to a quartic equation in the unknown loading. Optimum solutions based on numerical solution of the quartic are given in Ref. [6] for the cases of hover and climb—but not for descent. In this paper, we solve this quartic equation to find the optimum loading for a powered rotor in descent and then compute the wake contraction due to this loading. It will be shown that the optimum loading decreases to zero as the rotor approaches the wind-turbine vortex-ring boundary such that vortex-ring state is not encountered when the rotor is optimized.

Glauert's second approximation to momentum theory implies that the induced flow at the rotor disk is parallel to the local thrust vector[4]:

$$\tan\phi = \frac{\omega x/2}{u} = \frac{U+u}{\Omega x - \omega x/2} \tag{1}$$

where ϕ is the angle of inflow and thrust vector, ω the wake rotational speed, x the radial coordinate, u the induced flow at rotor, U the climb rate and Ω the rotor singular speed. From Eq. (1), u can be written in terms of ω (or vice versa). The incremental thrust $\mathrm{d}T$ and incremental power $\mathrm{d}P$ at a radial station thus become:

$$\mathrm{d}T = 2\pi\rho\left[\left(\Omega - \frac{\omega}{2}\right)\omega x^3\right]\mathrm{d}x \tag{2}$$

$$\begin{aligned}\mathrm{d}P &= 2\pi\rho\left[\left(\Omega - \tfrac{\omega}{2}\right)\Omega\omega x^4\right]\tan\phi\,\mathrm{d}x\\ &= 2\pi\rho[(U+u)\Omega\omega x^3]\mathrm{d}x\end{aligned} \tag{3}$$

One can adjoin the thrust to the power with a Lagrange multiplier Λ and obtain a variational statement for the minimum induced power P_{I} given a specified thrust.

$$\begin{cases} P_{\mathrm{I}} = P - UT \\ J = P_{\mathrm{I}} - T\Lambda \\ \delta(J) = 0 \end{cases} \tag{4}$$

where J is the power functional and $\delta(\cdot)$ the variational operator. This variational statement for power results in a quartic equation for the optimum angular velocity ω at any radial location—from which one can find the optimum u and the optimum loading, $\mathrm{d}T/\mathrm{d}\bar{r}$[6]:

$$\begin{aligned}&\left[\left(1+3q-q^2\right)X - 2(2+2q-q^2)\right]^2\left[(1-q)^2X^2 + 4(X-1)\bar{r}^2\right]\\ &\quad = \left[(1-q)^2X^2 + 2\bar{r}^2(3X-4)\right]^2\end{aligned} \tag{5}$$

with

$$\begin{cases} X = \frac{2\Omega}{\omega} \\ q = \frac{v_0}{\eta+v_0} \\ \bar{r} = \frac{x}{R(\eta+v_0)} \end{cases} \tag{6}$$

where R is the rotor radius, q the normalized loading parameter, q the nondimensional climb rate ($\eta = U/(\Omega R)$) and v_0 the nondimensional Lagrange multiplier—which becomes the Glauert loading parameter. Once Eq. (5) is solved for $X(\bar{r})$, Eq. (1) can be used to find ω, u, and the bound circulation of the optimum loading Γ:

$$\begin{cases} \overline{\omega} = \frac{\omega}{\Omega} \\ \frac{\Gamma}{2\pi\Omega R^2(\eta+v_0)^2} = \overline{\omega}\bar{r}^2 \\ \bar{u} = \frac{u}{\Omega R(\eta+v_0)} = -\frac{1-q}{2} + \left[\frac{(1-q)^2}{4} + \left(1-\frac{\overline{\omega}}{2}\right)\left(\frac{\overline{\omega}}{2}\right)\bar{r}^2\right]^{1/2} \end{cases} \tag{7}$$

Therefore, solution of the quartic Eq. (5) gives the entire solution for an optimum rotor in climb, hover, or descent. The optimum thrust and power are determined by Eqs. (2) and (3).

$$\frac{\mathrm{d}C_T}{\mathrm{d}\bar{r}} = (\eta+v_0)^4(2\overline{\omega} - \overline{\omega}^2)\bar{r}^3 \tag{8}$$

$$\frac{\mathrm{d}C_P}{\mathrm{d}\bar{r}} = (\eta+v_0)^5(1-q+\bar{u})(\overline{\omega})\bar{r}^3 \tag{9}$$

where C_T is the thrust coefficient $(T/(\rho\pi R^2\Omega^2R^2))$ and C_P is the power coefficient $(P/(\rho\pi R^2\Omega^3R^3))$. Eqs (5)–(9) are sufficient to describe the optimum rotor in hover, climb, or descent under Glauert's second approximation to momentum theory. For hover, $q = 1$; for climb, $0 < q < 1$; and, for descent, $q > 1$. Thus, the above equations give a normalized form of the optimum rotor for all powered states.

2. Solution method

For hover ($\eta = 0$, $q = 1$), Eq. (4) reduces to a cubic in X which can be solved in closed form for the unknown X and, consequently, for ω. As shown in Ref. [3], that cubic has a compact closed form solution.
where

$$\overline{\omega} = \frac{6}{5+\bar{r}^2+2(1+\bar{r}^2)\cos(\theta/3)} \tag{10}$$

$$\theta = \arccos\frac{\bar{r}^6+3\bar{r}^4+3\bar{r}^2-1}{\bar{r}^6+4\bar{r}^4+3\bar{r}^2+1} \tag{11}$$

For rotors in climb or descent, one must deal with the entire quartic in Eq. (5). Although there is a closed form solution to the quartic, it is quite cumbersome. For computational purposes, the most efficient approach is to solve Eq. (5) numerically for any given value of q and $\bar{r}$. There are four numerical roots for each case, but the physically meaningful root is always the smallest, positive-real value for $\overline{\omega}$.

It will be interesting to compare this optimum solution with the Betz loading, the latter of which can be expressed as:

$$\overline{\omega} = \frac{2q_{\mathrm{B}}}{1+\bar{r}^2} \tag{12}$$

where q_{B} is the Betz loading parameter. Later, we will make this comparison. However, it should first be noted that, in the Betz loading in Eq. (12), the loading parameter q_{B} is based on the Betz loading variable v_0, $q_{\mathrm{B}} = v_0/(\eta+v_0)$, where for Betz the parameter v_0 is equal to v_∞ (the induced flow for large $\bar{r}$). In contrast, the parameter v_0 used to define q and $\bar{r}$ in Eqs. (5) and (6) is only equal to the far-field induced flow for $q = 0$ and $q = 1$. The parameter v_0 varies slightly from v_∞ in the range $0 < q < 1$ and varies substantially for $q > 1.5$. Later, we will give the exact correspondence between the values of q_{B} and q for Betz and for the Glauert optimum.

Either the optimum or the Betz solution can be placed into Eqs. (7)–(9) to obtain the loading and inflow. For small q, the Glauert solution approaches the Betz solution. Note, also, that