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# Flutter analysis of an airfoil with nonlinear damping using equivalent linearization

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**Abstract** The equivalent linearization method (ELM) is modified to investigate the nonlinear flutter system of an airfoil with a cubic damping. After obtaining the linearization quantity of the cubic nonlinearity by the ELM, an equivalent system can be deduced and then investigated by linear flutter analysis methods. Different from the routine procedures of the ELM, the frequency rather than the amplitude of limit cycle oscillation (LCO) is chosen as an active increment to produce bifurcation charts. Numerical examples show that this modification makes the ELM much more efficient. Meanwhile, the LCOs obtained by the ELM are in good agreement with numerical solutions. The nonlinear damping can delay the occurrence of secondary bifurcation. On the other hand, it has marginal influence on bifurcation characteristics or LCOs.



## 1. Introduction

Nonlinear airfoil flutter is a typical self-excited vibration with rich nonlinear dynamical behaviors, such as limit cycle oscillation (LCO), bifurcation, and chaos.[1–4] Since not all the nonlinear features can be predicted by numerical methods, lots of analytic or semi-analytic techniques have been applied on airfoil models, for example, the harmonic balance method (HBM),[5,6] the incremental harmonic balance (IHB) method,[7,8] the perturbation-incremental method,[9] the homotopy analysis method,[10] and the equivalent linearization method (ELM),[11–14] to mention a few.

The ELM has been widely applied to various nonlinear vibration problems due to its simplicity and effectiveness. Furthermore, the approximate solution of the equivalent linear system has clear physical significance, which can provide us with convenience to analyze nonlinear dynamical behaviors. One of the most important procedures of the ELM is to derive an equivalent linear system by linearizing considered nonlinearities. Usually, the average method or the KBM method is employed for obtaining equivalent linear quantities.[3] Based on an equivalent linear system, methods for linear flutter analysis can be applied. For example, Liu and Zhao[1] gained the equivalent stiffness for cubic pitching nonlinearity by the average method. Later, Mickens[11] proposed a method by combining equivalent linearization and the averaging technique. Lim and Wu[12] combined the ELM and the HBM for solving strongly nonlinear vibration. Chen and Liu[13] improved the accuracy of equivalent stiffness by Lim's method to analyze the influences of quadratic pitching stiffness on a flutter system. Most recently, the ELM was extended to flutter systems with multiple nonlinearities, as suggested by Chen et al.[14]

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Structural nonlinearities such as cubic pitch/plunge stiffness, freeplay, and hysteresis have been extensively investigated in nonlinear airfoil flutter. Nonlinear damping, however, has rarely been investigated. Note that nonlinear damping may arise in hinge moment, damper, or solid friction.[15–18] Nonlinear damping may also play a considerable part in the behavior of nonlinear systems, especially strongly coupled fluid–structure systems.[15] Via an experiment for tube arrays, Meskell and Fitzpatrick[16] pointed out that the resulted self-excited LCO amplitude was determined by nonlinear damping while linear damping dominated the instability flutter. After that, a method was suggested by Meskell[17] for estimating the damping parameters in lightly damped systems. The influences of damping on limit cycles were also described by Sinou and Jezequel.[18] To the best of our knowledge, very few investigations addressed the effect of nonlinear damping on airfoil flutter.

This study aims at extending the ELM to investigate the flutter system of an airfoil with structural nonlinear damping. Special emphasis is put on the effectiveness of the ELM and the influences of nonlinear damping on LCOs. The equivalent linearization quantity of nonlinear damping is obtained by the average method. The LCO frequency is chosen as an active increment to produce bifurcation charts. Then, the LCOs and bifurcation of the equivalent flutter system are analyzed in detail. Numerical examples validate the accuracy of the extended ELM.

## 2. Equations of motions

Fig. 1 shows the physical model of a two-dimensional airfoil, which oscillates in the pitch and plunge directions. The plunge deflection is denoted by $\bar{h}$, positive if downward; the symbol $\alpha$ denotes the pitch angle, positive if nose up. The length of the mid-chord is $b$. The mass center (c.g.) resides at a distance $x_\alpha b$ from the elastic axis ($E$). Besides, there is a distance $a_h b$ between the elastic axis and the mid-chord. The focal point $F$ is the aerodynamic center. Both of the distances are positive when measured towards the trailing edge of the airfoil.

The coupled equations for the motions of the airfoil subject to subsonic aerodynamics can be modeled in a non-dimensional form as follows[4,19]

$$\begin{cases} \ddot{\xi}+x_\alpha\ddot{\alpha}+2\varsigma_\xi\overline{\omega}\dot{\xi}/U+(\overline{\omega}/U)^2G(\xi)=-C_L(t)/\pi\mu+P(t)b/mU^2 \\ x_\alpha\ddot{\xi}/r_\alpha^2+\ddot{\alpha}+2\varsigma_\alpha\dot{\alpha}/U+(1/U)^2M(\alpha)=2C_M(t)/\pi\mu r_\alpha^2+Q(t)b/mU^2r_\alpha^2 \end{cases} \tag{1}$$

where $\xi=\bar{h}/b$ is the non-dimensional displacement, and the prime denotes the differentiation with respect to the non-dimensional time $t$, which is defined as $t = Ut_1/b$ ($t_1$ is the real time and $U$ is a non-dimensional flow velocity given by

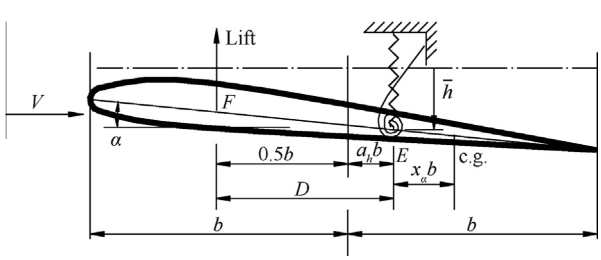


**Fig. 1** Physical model of a two-dimensional airfoil.

$U = V/b\omega_\alpha$ with $V$ as the flow speed); $\overline{\omega}$ is indicated by $\overline{\omega}=\omega_\xi/\omega_\alpha$, where $\omega_\xi$ and $\omega_\alpha$ are the uncoupled natural frequencies in the plunge and pitch modes, respectively; $\varsigma_\xi$ and $\varsigma_\alpha$ are the damping ratios; $G(\xi)$ and $M(\alpha)$ denote the nonlinear terms of plunging and pitching, respectively; $P(t)$ and $Q(t)$ are the externally applied force and moment; $r_\alpha$ is the radius of gyration about the elastic axis; $m$ is the airfoil mass per unit length while $\mu$ is the airfoil-air mass ratio. $C_L(t)$ and $C_M(t)$ denote the coefficients for lifting and moment, respectively. For an incompressible flow, $C_L(t)$ and $C_M(t)$ can be modeled by

$$\begin{cases} C_L(t)=\pi\left(\ddot{\xi}-a_h\ddot{\alpha}+\dot{\alpha}\right)+2\pi\left\{\alpha(0)+\dot{\xi}(0)+(1/2-a_h)\dot{\alpha}(0)\right\}\varphi(\tau) \\ \qquad +2\pi\int_0^t\varphi(t-\sigma)\left\{\dot{\alpha}(\sigma)+\ddot{\xi}(\sigma)+(1/2-a_h)\ddot{\alpha}(\sigma)\right\}\mathrm{d}\sigma \\ C_M(t)=\pi(1/2+a_h)\left\{\alpha(0)+\dot{\xi}(0)+(1/2-a_h)\dot{\alpha}(0)\right\}\varphi(\tau) \\ \qquad +\pi(1/2+a_h)\int_0^t\varphi(t-\sigma)\left\{\dot{\alpha}(\sigma)+\ddot{\xi}(\sigma)+(1/2-a_h)\ddot{\alpha}(\sigma)\right\}\mathrm{d}\sigma \\ \qquad +\pi a_h\left(\ddot{\xi}-a_h\ddot{\alpha}\right)/2-(1/2-a_h)\frac{\pi}{2}\dot{\alpha}-\pi\ddot{\alpha}/16 \end{cases} \tag{2}$$

where the Wagner function $\varphi(\tau)$ is given by the Jone's approximation $\varphi(t) = 1-\psi_1\mathrm{e}^{-\varepsilon_1 t}-\psi_2\mathrm{e}^{-\varepsilon_2 t}$, with the constants $\psi_1 = 0.165$, $\psi_2 = 0.335$, $\varepsilon_1 = 0.0455$, and $\varepsilon_2 = 0.3$.[20]

Due to the existence of the integral terms in Eq. (2), Eq. (1) is a system of integro-differential equations. Studying the dynamic behavior of the system analytically can be rather cumbersome. Lee et al.[21] introduced four new variables for eliminating the integral terms

$$w_1 = \int_0^t \mathrm{e}^{-\varepsilon_1(t-\sigma)}\alpha(\sigma)\mathrm{d}\sigma, \quad w_2 = \int_0^t \mathrm{e}^{-\varepsilon_2(t-\sigma)}\alpha(\sigma)\mathrm{d}\sigma$$

$$w_3 = \int_0^t \mathrm{e}^{-\varepsilon_1(t-\sigma)}\xi(\sigma)\mathrm{d}\sigma, \quad w_4 = \int_0^t \mathrm{e}^{-\varepsilon_2(t-\sigma)}\xi(\sigma)\mathrm{d}\sigma$$

Thus, Eq. (1) can then be rewritten in a general form containing only differential operators as

$$\begin{cases} c_0\ddot{\xi}+c_1\ddot{\alpha}+c_2\dot{\xi}+c_3\dot{\alpha}+c_4\xi+c_5\alpha+c_6w_1+c_7w_2 \\ \quad +c_8w_3+c_9w_4+c_{10}G(\xi)=f(t) \\ d_0\ddot{\xi}+d_1\ddot{\alpha}+d_2\dot{\xi}+d_3\dot{\alpha}+d_4\xi+d_5\alpha+d_6w_1+d_7w_2+d_8w_3 \\ \quad +d_9w_4+d_{10}M(\alpha)=g(t) \end{cases} \tag{3}$$

where the coefficients $c_0, c_1, \cdots, c_{10}$ and $d_0, d_1, \cdots, d_{10}$ are given in Ref. 19. Both $f(t)$ and $g(t)$ depend on initial conditions, Wagner's function, and the external forcing terms,

$$\begin{cases} f(t)=\frac{2}{\mu}\left\{\left(\frac{1}{2}-a_h\right)\alpha(0)+\xi(0)\right\}(\psi_1\varepsilon_1\mathrm{e}^{-\varepsilon_1 t}+\psi_2\varepsilon_2\mathrm{e}^{-\varepsilon_2 t})+\dfrac{P(t)b}{mU^2} \\ g(t)=-\dfrac{1+2a_h}{2r_\alpha^2}f(t)+\dfrac{Q(t)}{mU^2r_\alpha^2} \end{cases}$$

Generally, the identification of the nonlinearities on the airfoil is very complicated. The nonlinear damping terms are usually assumed to be proportional to the cubic power of velocity.[22] In this study, we consider the system with cubic damping as

$$\begin{cases} G(\xi)=k_\xi\xi+e_\xi\dot{\xi}^3 \\ M(\alpha)=k_\alpha\alpha+e_\alpha\dot{\alpha}^3 \end{cases} \tag{4}$$

where $k_\xi$, $k_\alpha$, $e_\xi$ and $e_\alpha$ are all constants.

Introduce a variable vector $\boldsymbol{X}=[x_1 \quad x_2 \quad \cdots \quad x_8]^{\mathrm{T}}$ with $x_1 = \alpha$, $x_2 = \dot{\alpha}$, $x_3 = \xi$, $x_4 = \dot{\xi}$, $x_5 = w_1$, $x_6 = w_2$, $x_7 = w_3$