Commun Nonlinear Sci Numer Simulat 42 (2017) 106–120

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

Practical stability with respect to initial time difference for Caputo fractional differential equations

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ARTICLE INFO

Article history:
Received 19 January 2016
Revised 29 April 2016
Accepted 4 May 2016
Available online 10 May 2016

Keywords:
Practical stability
Different initial data
Lyapunov functions
Caputo fractional differential equations
Caputo fractional Dini derivative

ABSTRACT

Practical stability with initial data difference for nonlinear Caputo fractional differential equations is studied. This type of stability generalizes known concepts of stability in the literature. It enables us to compare the behavior of two solutions when both initial values and initial intervals are different. In this paper the concept of practical stability with initial time difference is generalized to Caputo fractional differential equations. A definition of the derivative of Lyapunov like function along the given nonlinear Caputo fractional differential equation is given. Comparison results using this definition and scalar fractional differential equations are proved. Sufficient conditions for several types of practical stability with initial time difference for nonlinear Caputo fractional differential equations are obtained via Lyapunov functions. Some examples are given to illustrate the results.

1. Introduction

Fractional calculus is the theory of integrals and derivatives of arbitrary non-integer order, which unifies and generalizes the concepts of ordinary differentiation and integration. For more details on geometric and physical interpretations of fractional derivatives and a general historical perspective, we refer the reader to the monographs [8,9,26] and the cited references therein. Fractional differential equations (FrDE) which involve fractional derivatives, arise frequently in a variety of branches of science and engineering problems; for example to describe the anomalous dynamics of various processes related to complex systems ([25]). There are two main approaches in the theoretical formulation of initial value problems for fractional differential equations. One of them is based on the interpretation of the initial condition of fractional systems as a distributed initial condition ([14,27,30], [31]) . The other is based on the property of the Caputo fractional derivative of a constant ([6,7,11,20]). This paper uses the second approach.

One of the main problems in the qualitative theory of differential equations is the stability of solutions. Many different types of stability are defined and studied in the literature. One of them is practical stability whose definition is based on the predefined boundaries for the perturbation of initial conditions response and the allowable perturbation of the system response (see, for example, [5,12,15,17,18], the book [21] and the cited therein references).

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http://dx.doi.org/10.1016/j.cnsns.2016.05.005
1007-5704/

Recently, most results for stability via Lyapunov functions in fractional differential equations could be divided into two main groups:

- Continuously differentiable Lyapunov functions with their Caputo derivatives among the solutions of the given fractional differential equation (see, for example, the papers [4,23]).
- Continuous Lyapunov functions with their fractional Dini fractional derivatives among the solutions of the given fractional differential equation (see, for example, the papers [11,20,22]).

Often in real situations it may be impossible to keep the initial time unchanged and to change only the space variable. This situation requires introducing and studying a new generalization of the classical concept of stability which involve a change in both the initial time and the initial values, so called stability with initial time difference. Various types of stability with initial time difference were studied for ordinary differential equations ([19,29,34]), functional differential equations ([1,16]), fractional differential equations ([7,32,33]). Note this stability is meaningful only in the nonautonomous case (for explanations and some examples, see [1]).

Recently, in [10] (Theorem 2.1) the classical initial value problem for Caputo-type fractional differential equations was studied in the case when the location of the starting point of the differential operator is changeable and the continuous dependence of the solution at this starting point was investigated. In this paper both initial time and initial values are changeable. We define in an appropriate way the practical stability with initial time difference as well as the uniform practical stability with initial time difference for nonlinear Caputo fractional differential equations.

Studying stability with initial time difference by Lyapunov functions for fractional differential equations requires a new definition of the derivative of these functions (and a new type of comparison equation containing a parameter). There are problems in Definition 2.9 in [32], Definitions 2.3.1, 2.3.2 and 2.3.5 in [33] and Definition 2.1 (DD2) in [7].

In the paper a new type of derivative of Lyapunov functions called the Caputo fractional Dini derivatives with initial time difference is defined. We show its advantages compared to what is known in the literature. With appropriate examples we show the natural relationship between the derivative of Lyapunov functions and the Caputo fractional Dini derivative used in the studied equations. Several sufficient conditions for practical stability with initial data difference for nonlinear fractional differential equations based on Lyapunov's functions and comparison results for a nonlinear scalar fractional differential equation with a parameter are obtained. The assumption for locally Holder continuity or C^q continuity in the literature are weakened to just continuity. Some examples are given to illustrate the main results.

2. Notes on fractional calculus

Fractional calculus generalizes the derivative and the integral of a function to a non-integer order [8,9,26] and there are several definitions of fractional derivatives and fractional integrals.

In many applications in science and engineering, the fractional order q is often less than 1, so we restrict $q \in (0, 1)$ everywhere in the paper.

1: The Riemann–Liouville (RL) fractional derivative of order $q \in (0, 1)$ of $m(t)$ is given by (see, for example, Section 1.4.1.1 [8], or [26]) ${}_{t_0}D^q m(t) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int_{t_0}^t (t-s)^{-q} m(s)ds,\ t \geq t_0$ where $\Gamma(.)$ denotes the Gamma function.

2: The Caputo fractional derivative of order $q \in (0, 1)$ is defined by (see, for example, Section 1.4.1.3 [8])

$$ {}_{t_0}^{c}D^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s)ds, \qquad t \geq t_0. \tag{1} $$

The Caputo and Riemann–Liouville formulations coincide when the initial conditions are zero. Note, the RL derivative is meaningful under weaker smoothness requirements.

3: The Grünwald–Letnikov fractional derivative is given by (see for example, Section 1.4.1.2 [8])

$$ {}_{t_0}\widetilde{D}_t^q m(t) = \lim_{h\to 0} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh), \qquad t \geq t_0 $$

and the Grünwald–Letnikov fractional Dini derivative by

$$ {}_{t_0}\widetilde{D}_+^q m(t) = \limsup_{h\to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r \binom{q}{r} m(t-rh), \qquad t \geq t_0, \tag{2} $$

where $\binom{q}{r} = \frac{q(q-1)\dots(q-r+1)}{r!}$ and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

For a wide class of functions, the definitions of the Grünwald–Letnikov fractional derivative and the Riemann–Liouville fractional derivative are equivalent (for example, if the functions are sufficiently smooth). This allows us to use Grünwald–Letnikov fractional derivative for the formulation of the problem and for proving the theoretical result. One could then turn to the Riemann–Liouville fractional derivative for applied problems.