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Research paper

An *hp* symplectic pseudospectral method for nonlinear optimal control

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ABSTRACT

An adaptive symplectic pseudospectral method based on the dual variational principle is proposed and is successfully applied to solving nonlinear optimal control problems in this paper. The proposed method satisfies the first order necessary conditions of continuous optimal control problems, also the symplectic property of the original continuous Hamiltonian system is preserved. The original optimal control problem is transferred into a set of nonlinear equations which can be solved easily by Newton–Raphson iterations, and the Jacobian matrix is found to be sparse and symmetric. The proposed method, on one hand, exhibits exponent convergence rates when the number of collocation points are increasing with the fixed number of sub-intervals; on the other hand, exhibits linear convergence rates when the number of sub-intervals is increasing with the fixed number of collocation points. Furthermore, combining with the *hp* method based on the residual error of dynamic constraints, the proposed method can achieve given precisions in a few iterations. Five examples highlight the high precision and high computational efficiency of the proposed method.

1. Introduction

As an important part of nonlinear science and engineering, the technique of nonlinear optimal control theories and methods has been widely used in diverse engineering fields, such as chemical engineering [1–3], vibration engineering [4,5], robotics [6–8] and astrodynamics [9–12]. It is extremely difficult to analytically solve a nonlinear optimal control problem. Thus numerous computational techniques for solving nonlinear optimal control problems have been developed in the literature, and they fall into two general categories [13,14]: direct methods and indirect methods.

In an indirect method, the original nonlinear optimal control problem is transformed into a nonlinear two-point boundary value problem (TPBVP) using the first order necessary conditions for optimal control problems [13]. The main advantages of indirect methods come down to two aspects. On one hand, once a numerical solution is obtained then it naturally becomes a locally optimal solution since first order necessary conditions are used. On the other hand, as the solution of most indirect methods includes the information of costate variables, together with the value of state variables it can be used to analyze the characteristic of geometric structures of the Hamiltonian dynamic system. However, different kinds of constrains are required to be handled specifically according to the type of constraints. Various numerical methods such as the shooting method [15–17], the multiple shooting method [18], the generating function method [19] and the finite difference method [20] can be adopted to solve TPBVPs. For shooting methods, initial guesses of unknown state and/or costate variables which

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must satisfy transversality conditions are required. They can achieve effective convergence but may cause ill-conditioning for problems with a long time interval. In generating function methods, guesses of initial or terminal costate variables are not needed which is an advantage over the shooting method. However, complicated series expansions and lots of ordinary differential equations are required in generating function methods.

In contrast to indirect methods, direct methods try to optimize the cost functional directly. In a direct method, the nonlinear optimal control problem is transformed into a finite dimensional nonlinear programming problem (NLP) [13]. Various types of constraints can be treated in a uniform framework [21] in direct methods which is an advantage over indirect methods. Over past two decades, direct collocation nonlinear programming program (DCNLP) methods have become popular. In DCNLP methods, state and control variables are approximated by a set of trial functions and the dynamic constraints are collocated at specified set of points in the solution domain. More recently, pseudospectral methods have attracted much attention. In most pseudospectral methods, the collocation points are based on Gaussian quadrature rules and the basic functions are commonly Legendre or Chebyshev polynomials. According to the scheme of interpolation, various kinds of interpolation points such as Chebyshev–Gauss–Lobatto (CGL) [22], Legendre–Gauss (LG) [23,24], Legendre–Gauss–Radau (LGR) [25–28] and Legendre–Gauss–Lobatto (LGL) [29] can be obtained. Estimations of costate variables cannot be obtained in the majority of direct methods so that it is impossible to determine whether those methods satisfy the first order necessary conditions for optimal control problems. Nevertheless, it has been proved that the Karush–Kuhn–Tucker (KKT) conditions in pseudospectral methods are equivalent to the first order necessary conditions for optimal control problems. For problems whose solutions are smooth and well behaved, pseudospectral methods have a simple structure and converge at an exponential rate. However, the exponential convergence rate may lose when solutions are non-smooth. Due to these advantages, pseudospectral methods have been widely used in aeronautics and astronautics and a software package, i.e., GPOPS [26] based on MATLAB is developed. More recently, Mardsen and his coworkers proposed discrete mechanics and optimal control (DMOC) [30] which belongs to direct methods using discretization. In DMOC state equations are discretized based on the Lagrange–d' Alembert principle, which plays an important role in theoretical mechanics.

The optimal control problems can be transferred into Hamiltonian system by the variational principle or the Pontryagin's maximum principle. The most fundamental property of Hamiltonian systems is that the phase flow is a symplectic transformation [31]. Thus numerical methods that preserve the symplectic structure are more effective and accurate for solving Hamiltonian systems. In fact, generating function methods and the DMOC method mentioned above are both symplectic algorithms. Recent years, Peng and his coworkers proposed a series of symplectic methods where state and costate variables within a time interval are approximated by using Lagrange polynomial and variables at two time ends of the time interval are taken as independent variables [32–36]. According to different choices of independent variables, four different kinds of action are proposed.

Originally, the DCNLP methods mentioned above are developed as *h* methods, where the whole time domain is divided into some small sub-intervals and state and/or control variables are approximated using fixed order of interpolation function in each sub-interval. Then, convergences in *h* methods are achieved by increasing the number of sub-intervals and/or replacing the position of sub-intervals. While primarily, pseudospectral methods are considered to be *p* methods, where the approximation of state and/or control variables are implemented in the whole time domain which is treated as a single mesh interval. Convergences in *p* methods are thus accomplished by increasing the degree of approximation polynomial. Yet, both *h* methods and *p* methods are born with limitations. Specifically, it may lead to an extremely fine mesh (in *h* methods) or an unreasonably high degree of interpolation function (in *p* methods) for achieving the given accuracy. In order to combine the characteristic of *h* methods and *p* methods, *hp* methods, where both the number of sub-intervals and the degree of approximation polynomials are allowed to change, have been developed [24,27,28,37]. *Hp* methods are originally developed for solving partial differential equations [38–41] and applied to solving optimal control problems in recent years. In this paper, an *hp* adaptive method based on the residual error of dynamic equations is developed.

Since symplectic-preserving algorithms are highly efficient, and pseudospectral methods exhibit the exponent rate of convergence for problems where solutions are smooth, a symplectic pseudospectral method is attractive. Thus, an adaptive symplectic pseudospectral method for solving nonlinear optimal control problems is proposed in this paper. The original optimal control problem is transformed into a nonlinear TPBVP using the first order necessary conditions for optimal control problems. The whole time domain is divided into several sub-intervals, and the action within each sub-interval is approximated by the LGL quadrature scheme. Thus state and costate variables are naturally discretized at LGL points and approximated by Lagrangian interpolation functions. Based on the dual variational principle and taking state variables at two ends of a sub-interval as independent variables, the nonlinear TPBVP is transformed into a set of nonlinear equations. After applying boundary conditions, the set of nonlinear equations can be solved easily by Newton–Raphson iteration methods. The Jacobian matrix in the Newton iterations is found to be sparse and symmetric, and then lead to a high computational efficiency and high precision. Initial guesses in the Newton iterations are not required to satisfy transversality conditions and final solutions are not sensitive to initial guesses. Combining with an *hp* adaptive procedure based on residual error of dynamic constraints, numerical solutions of given precision can be obtained in a few iterations.

2. Problem formulation

Considering the following nonlinear optimal control problem, the main objective of optimal control is to determine the state variables $x \in \mathbb{R}^d$ and the control input $\mathbf{u} \in \mathbb{R}^p$ within the time interval $t \in [0, t_f]$ to minimize the following Bolza cost