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Research paper

On fractional Langevin equation involving two fractional orders

Omid Baghani

Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran

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ABSTRACT

In numerical analysis, it is frequently needed to examine how far a numerical solution is from the exact one. To investigate this issue quantitatively, we need a tool to measure the difference between them and obviously this task is accomplished by the aid of an appropriate norm on a certain space of functions. For example, Sobolev spaces are indispensable part of theoretical analysis of partial differential equations and boundary integral equations, as well as are necessary for the analysis of some numerical methods for the solving of such equations. But most of articles that appear in this field usually use $\|.\|_\infty$ in the space of $C[a, b]$ which is very restrictive. In this paper, we introduce a new norm that is convenient for the fractional and singular differential equations. Using this norm, the existence and uniqueness of initial value problems for nonlinear Langevin equation with two different fractional orders are studied. In fact, the obtained results could be used for the classical cases. Finally, by two examples we show that we cannot always speak about the existence and uniqueness of solutions just by using the previous methods.

1. Introduction

In 1908, the French physicist Paul Langevin proposed an elaborate description of Brownian motion, that is the random movement of a particle submerged in a fluid, due to its collisions with the much smaller fluid molecules. Since then, Langevin was considered as one of the institutors of the branch of stochastic differential equations [1]. Many of stochastic problems in fluctuating environments are described by Langevin equation [2]. But for some complex systems, the classical Langevin equation cannot offer a correct concept of the problem. As a result, various generalizations have been offered which make up the lacks of the classic case and they describe more physical phenomena in disordered regions [3]. Above all, Kubo [4,5], introduced the general Langevin equation (GLE) for modeling anomalous diffusive processes in complex and viscoelastic environment. Another generalization of the Langevin equation is obtained naturally by replacing the ordinary derivative by a fractional derivative which yields the well known fractional Langevin equation (FLE). The fractional Langevin equation was used for modeling of single-file diffusion [6] and for a free particle driven by power law type of noises [7]. In [8], the transformation of the Fokker-Planck equation, which corresponds to the Langevin equation with multiplicative white noise, into the Wiener process is made available for any prescription. Wang et al. [9] discussed Ulam-Hyers stability of nonlinear fractional Langevin equations using the boundedness, monotonicity and nonnegative properties of classical and generalized Mittag-Leffler functions. The references [10–12] introduce a complete and large picture of numerical simulation using the meshless method for fractional equation. The nonlinear Langevin equation with two fractional orders was introduced and

E-mail address: omid.baghani@gmail.com

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studied in [13–18]. We know that most of the nonlinear fractional equations cannot be analytically solved, so firstly the existence and uniqueness of solutions should be verified. Recently, the existence and uniqueness of solutions of the initial and boundary value problems for nonlinear fractional equations are extensively studied. For example, see Kosmatov [19], Deng et al. [20,21], Agarwal et al. [22], Zhou et al. [23] and the references therein [24–26].

In this paper, we discuss mainly the existence and uniqueness of solution for the initial value problem of Langevin equation of the following type:

$$\begin{cases} D^{\beta}(D^{\alpha}+\gamma)x(t)=f(t,x(t)), & 0<t\le 1,\\ x^{(k)}(0)=\mu_k, & 0\le k<l,\\ x^{(\alpha+k)}(0)=\nu_k, & 0\le k<n, \end{cases} \tag{1}$$

where $\gamma\in\mathbb{R}$, $m-1<\alpha\le m$, $n-1<\beta\le n$, $l=\max\{m,n\}$, $m,n\in\mathbb{N}$, D^{α} and D^{β} are the Caputo fractional derivatives and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a Lebesgue measurable function.

Our work was motivated by the work of Tao Yu et al. [13]. In obtained results in that paper, for the existence, the nonlinear term f needs to satisfy the following two conditions:

(i) $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuously differentiable function,
(ii) there exist nonnegative functions $a_1, a_2\in C[0,1]$ with

$$A=\sup_{0\le t\le 1}\left(\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}a_2(s)ds+\frac{|\gamma|}{\Gamma(\alpha+1)}\right)<1, \tag{2}$$

and

$$0<B=\sup_{0\le t\le 1}\left(\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}a_1(s)ds+|\phi(t)|\right)<\infty, \tag{3}$$

where

$$\phi(t)=\sum_{i=0}^{n-1}\frac{\nu_i+\gamma\mu_i}{\Gamma(\alpha+i+1)}t^{\alpha+i}+\sum_{j=0}^{m-1}\frac{\mu_j}{\Gamma(j+1)}t^{j}, \tag{4}$$

such that for each $0\le t\le 1$ and $z\in\mathbb{R}$,

$$|f(t,z)|\le a_1(t)+a_2(t)|z|.$$

For the uniqueness, the nonlinear term f needs to satisfy the condition (i) and the following condition:
(iii) there exists nonnegative function $a\in C[0,1]$ with

$$\zeta=\sup_{0\le t\le 1}\left(\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}a(s)ds+\frac{|\gamma|}{\Gamma(\alpha+1)}\right)<1, \tag{5}$$

and

$$0<\eta=\sup_{0\le t\le 1}\left|\int_0^t \frac{(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}f(s,0)ds+\phi(t)\right|<\infty, \tag{6}$$

such that for each $0\le t\le 1$ and $z_1,z_2\in\mathbb{R}$,

$$|f(t,z_1)-f(t,z_2)|\le a(t)|z_1-z_2|.$$

Authors, explicitly assumed the continuity of $a(\cdot)$ while in our work, $a(\cdot)$ is allowed to be only measurable. It seems, this weaker assumption makes the new results to cover the case of one-sided Lipschitz continuity, therefore these results are stronger than those of [13]. Here, we emphasize on measurability of $a(\cdot)$ and we mainly use weighted spaces [27,28]. Also, the condition $B>0$ in (3) can be omitted because the equality $B=0$ means that a_1 and ϕ are equivalent to 0. The term $\phi(t)$ in this inequality can be omitted also, because the function ϕ is evidently bounded. The condition (6) is also can be simplified.

The classical Langevin formula for a Brownian particle is described by

$$m\frac{d^2x(t)}{dt^2}+\gamma\frac{dx(t)}{dt}=f(t,x(t)), \tag{7}$$

where m is the particle mass, γ is the coefficient of viscosity, x is the particle position and f is the force acting on the particle from molecules of the fluid encircling the Brownian particle [29–34]. Langevin suggested that the force f can be written as a sum of two parts. The first one is the frictional force which is proportionate to the particle velocity, $f_v(t)=-\frac{m}{\sigma}v(t)$, where $1/\sigma$ is the friction coefficient for unit mass and the function v is assumed to be continuous. The second one, arising from rapid thermal fluctuation, is regarded as random and independent of the motion of the particle. This part is called the random force and is denoted by $r(t)$. The random force $r(t)$ often depends on the Dirac delta distribution that can be viewed as a limit of Gaussian probability density function. By ignoring the random force $r(t)$ and by fixing $\alpha=1$ and $\beta=1$, the classical Langevin (7) is a special case of the problem (1).