Commun Nonlinear Sci Numer Simulat 42 (2017) 699–708

Contents lists available at ScienceDirect

Commun Nonlinear Sci Numer Simulat

journal homepage: www.elsevier.com/locate/cnsns

Research paper

On a nonlocal modified Korteweg-de Vries equation: Integrability, Darboux transformation and soliton solutions

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ARTICLE INFO

Article history:
Received 2 January 2016
Revised 20 April 2016
Accepted 9 June 2016
Available online 9 June 2016

2010 MSC:
37K10
37K40

Keywords:
Nonlocal modified Korteweg-de Vries equation
Integrability
Darboux transformation
Soliton solutions

ABSTRACT

Very recently, Ablowitz and Musslimani introduced a new integrable nonlocal nonlinear Schrödinger equation. In this paper, we investigate an integrable nonlocal modified Korteweg-de Vries equation (mKdV) which can be derived from the well-known AKNS system. We construct the Darboux transformation for the nonlocal mKdV equation. Using the Darboux transformation, we obtain its different kinds of exact solutions including soliton, kink, antikink, complexiton, rogue-wave solution, and nonlocalized solution with singularities. It is shown that these solutions possess new properties which are different from the ones for mKdV equation.

1. Introduction

Parity-time-symmetry (PT-symmetry) is an interesting topic in quantum mechanics. Bender and Boettcher showed [1] that in the spectrum of the Hamiltonian, the PT-symmetry has significant impact. They proved that large amounts of non-Herimitan Hamiltons with PT-symmetry possess real and positive spectrum. In general, a non-Hermitian Hamiltonian $H = \partial_{xx} + V(x)$ is called PT-symmetric if $V(x)$ holds for $V(x) = V^*(-x)$. If set $V(x,t) = p(x,t)p^*(-x,t)$ in the Hamiltonian H above, then the Schrödinger equation $ip_t = Hp$ is PT-symmetric. In recent years, many works on PT-symmetry have been presented [2–4]. PT-symmetry has been widely applied to many areas of physics, such as optics [4–6], quantum chromodynamics [7] and Bose-Einstein condensates [8], etc.

The well-known nonlinear Schrödinger equation (NLS)

$$iq_t(x,t) = q_{xx}(x,t) \pm 2|q(x,t)|^2 q(x,t), \tag{1}$$

which is PT-symmetric, has been widely studied since the significant work of Zakharov and Shabat [9]. The NLS equation plays an important role in many physical fields, including nonlinear optics [10], deep water waves [11], plasma physics [12] as well as in a purely mathematical context like differential geometry of curves [13]. Very recently, Ablowitz and Musslimani introduced a nonlocal NLS equation [14]:

$$iq_t(x,t) = q_{xx}(x,t) \pm 2q(x,t)q^*(-x,t)q(x,t), \tag{2}$$

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http://dx.doi.org/10.1016/j.cnsns.2016.06.015
1007-5704/

which is derived from the well-known AKNS system. Like the NLS equation (1), the nonlocal NLS Eq. (2) is also PT-symmetric. It is an integrable system with the Lax pair, infinitely many conservation laws and it can be solved by the inverse scattering transformation [14]. Also, interactions of the dark and antidark soliton have been studied through the Darboux transformation (DT) method [15]. Eq. (2) has different behaviors from Eq. (1). For instance, Eq. (2) simultaneously has both bright and dark soliton [16] and has nonlocal solutions with periodic singularities [14].

The famous modified Korteweg-de Vries equation (mKdV) reads

$$q_t(x,t) + 6q^2(x,t)q_x(x,t) + q_{xxx}(x,t) = 0. \tag{3}$$

Eq. (3) can be derived from Euler equation and has applications in fluid dynamical systems, plasma physics and other physical contexts [17–19]. There are many works on Eq. (3) in the last decades [20–22]. Specially, Wadati used inverse scattering transformation to investigate mKdV equation and achieved exact solutions, including N-solitons, multiple-pole solutions and solutions derived from PT-symmetric potentials [23–25]. Moreover, Hirota obtained N-solitons by bilinear technique and investigated multiple collisions of solitons [26]. The complexitons and shock solitons of Eq. (3) are constructed by Darboux transformation [27,28].

In this paper, motivated by the idea of Ablowitz and Musslimani in [14], we propose and investigate the following nonlocal mKdV equation:

$$q_t(x,t) + 6q(x,t)q(-x,-t)q_x(x,t) + q_{xxx}(x,t) = 0. \tag{4}$$

This Eq. (4) can be yielded from AKNS system by a reduction. This means that Eq. (4) is Lax integrable. In this paper, we will construct its Darboux transformation and its different kinds of exact solutions including soliton, kink, antikink, complexiton, rogue-wave solution, and nonlocalized solution with singularities. It will be shown that these solutions possess new properties which are different from the ones of classical mKdV equation.

2. Darboux transformation for the nonlocal mKdV equation

It has been demonstrated that Darboux transformation is a important method to achieve exact solutions for integrable nonlinear systems, including soliton, breather and rogue wave. In this section, we will construct the Darboux transformation for the integrable nonlocal mKdV Eq. (4). First of all, we show that integrable nonlocal mKdV Eq. (4) can be derived from the well-known AKNS system [29] by a reduction. In fact, the AKNS system relates to the following linear spectral problem:

$$\varphi_x = \mathbf{U}\varphi = \begin{pmatrix} -i\lambda & q(x,t) \\ r(x,t) & i\lambda \end{pmatrix}\varphi, \quad \varphi_t = \mathbf{V}\varphi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\varphi, \tag{5}$$

where $\varphi = (\varphi_1(x,t), \varphi_2(x,t))^{\mathrm{T}}$, and λ is the spectral parameter. Expanding A, B, and C as following:

$$A = a_3\lambda^3 + a_2\lambda^2 + \left(a_1 + \frac{a_3}{2}qr\right)\lambda - \frac{i}{4}a_3(qr_x - q_xr) + \frac{a_2}{2}qr + a_0,$$
$$B = ia_3q\lambda^2 + \left(ia_2q - \frac{a_3}{2}q_x\right)\lambda + \frac{i}{4}a_3(2q^2r - q_{xx}) - \frac{a_2}{2}q_x + ia_1q,$$
$$C = ia_3r\lambda^2 + \left(ia_2r + \frac{a_3}{2}r_x\right)\lambda + \frac{i}{4}a_3(2qr^2 - r_{xx}) + \frac{a_2}{2}r_x + ia_1r,$$

where $a_j\ (j = 0, 1, 2, 3)$ are constants. Then the compatibility condition of Eq. (5) $\mathbf{U}_t - \mathbf{V}_x + [\mathbf{U}, \mathbf{V}] = 0$ yields the following equation system:

$$\begin{aligned} q_t &= -\frac{ia_3}{4}(q_{xxx} - 6qrq_x) - \frac{a_2}{2}(q_{xx} - 2q^2r) + ia_1q_x + 2a_0q, \\ r_t &= -\frac{ia_3}{4}(r_{xxx} - 6qrr_x) + \frac{a_2}{2}(r_{xx} - 2qr^2) + ia_1r_x - 2a_0r. \end{aligned} \tag{6}$$

Set $a_0 = a_1 = a_2 = 0$, $a_3 = -4i$, and a symmetry reduction,

$$r(x,t) = -q(-x,-t), \tag{7}$$

integrable nonlocal mKdV Eq. (4) is derived from equation system (6). Similarly to the construction of Darboux transformation for classical integrable mKdV equation [30,31], we can obtain the Darboux transformation of integrable nonlocal mKdV Eq. (4) by the following procedure. Take the gauge transformation,

$$\varphi^{[1]} = \mathbf{T}^{[1]}\varphi. \tag{8}$$

Then, spectral problem (5) becomes

$$\varphi_x^{[1]} = (\mathbf{T}_x^{[1]} + \mathbf{T}^{[1]}\mathbf{U})\left(\mathbf{T}^{[1]}\right)^{-1}\varphi^{[1]} \triangleq \mathbf{U}^{[1]}\varphi^{[1]}, \quad \varphi_t^{[1]} = (\mathbf{T}_t^{[1]} + \mathbf{T}^{[1]}\mathbf{V})\left(\mathbf{T}^{[1]}\right)^{-1}\varphi^{[1]} \triangleq \mathbf{V}^{[1]}\varphi^{[1]}. \tag{9}$$

We will determine $\mathbf{T}^{[1]}$ such that $\mathbf{U}^{[1]}$ and $\mathbf{V}^{[1]}$ have the same form as $\mathbf{U}$ and $\mathbf{V}$, except substituting new potentials $q^{[1]}$, $r^{[1]}$ for old potentials q, r. Let

$$\mathbf{T}^{[1]} = \lambda\mathbf{I} + \mathbf{B}^{[1]}, \tag{10}$$