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Research paper

The closed-form solution of the reduced Fokker–Planck–Kolmogorov equation for nonlinear systems

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ABSTRACT

In this paper, we propose a new method to obtain the closed-form solution of the reduced Fokker–Planck–Kolmogorov equation for single degree of freedom nonlinear systems under external and parametric Gaussian white noise excitations. The assumed stationary probability density function consists of an exponential polynomial with a logarithmic term to account for parametric excitations. The undetermined coefficients in the assumed solution are computed with the help of an iterative method of weighted residue. We have found that the iterative process generates a sequence of solutions that converge to the exact solutions if they exist. Three examples with known exact steady-state probability density functions are used to demonstrate the convergence of the proposed method.

1. Introduction

Dynamic loadings on structures such as those due to winds, sea waves, earthquake, roadway irregularities, rocket are random in nature. These loadings can be strong enough to cause nonlinear responses of the structure. When the excitations are modeled as the Gaussian white noise, the response of the structure forms a Markov process whose response statistics is completely characterized by the probability density function governed by the Fokker–Planck–Kolmogorov (FPK) equation. There had been strong interests in finding solutions of FPK equations for nonlinear systems. The major progresses were well documented in the seminal book by Cai and Lin [1]. The book contains a very comprehensive selection of nonlinear stochastic systems for which the exact solutions of the reduced FPK equation had been obtained. There have been much less results of exact solutions reported since then. This paper presents an iterative method that can construct highly accurate solutions of the reduced FPK equations, and converge to the exact solutions of the reduced FPK equation for all the cases reported in [1].

Researchers have long had a strong interest in finding the exact solutions of the FPK equation. The exact solutions of the FPK equation can be found for a very few one dimensional cases [2,3]. For the two- and higher-dimensional systems, the exact solutions of the reduced FPK equation have obtained under strict conditions by Caughey and Fai [4], Dimentberg [5], Yong and Lin [6], Cai and Lin [1], and Zhu and his coworkers [7,8]. The concept of detailed balance is a popular approach used to construct the exact solution of the reduced FPK equation for nonlinear systems under both additive and multiplicative white noise excitations. Indeed, many examples of the exact solutions can be found in the framework of the detailed

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balance. In 1990s, Lin and Cai [9] developed a systematic procedure using the probability potential and detailed balance of the generalized probability potential to obtain exact solutions of the reduced FPK equation for single- and multi-degree-of-freedom nonlinear oscillatory systems. There has been no new progress on the exact solution of the reduced FPK equation since then.

Since there is a limited number of nonlinear stochastic systems for which we can obtain the exact solution of the FPK equation, we must develop procedures to compute the solution numerically or analytically.

In the literature, various approximate methods have been developed to solve the FPK equation, including the finite element method [10–15], finite difference method [16,17], path integral [18–20], iterative scheme [21,22], the principle of maximum entropy [23–25], eigenfunction-expansion [26–28], variational method [29] and method of weighted residue [30–36]. These methods have limitations. For example, the finite element and finite difference are computationally intensive. The path integration is feasible for the low dimensional systems. The solution obtained by the method of weighted residue is influenced by the weighting functions. More comments on these numerical methods were given in the review article [37] and a monograph [38].

In the last decade, Er proposed the method of exponential polynomial closure (EPC) to solve the reduced FPK equation [33–35]. In this method, the steady-state PDF is approximated as an exponential function of a polynomial in state variables with undetermined coefficients. The assumed solution satisfies the reduced FPK equation in the weak sense. The undetermined coefficients of the polynomial are determined by solving simultaneous quadratic algebraic equations derived from the method of weighted residue. Recently, the EPC method has been extended to the nonlinear systems excited by a Poisson white noise [39] or by a combined Gaussian and Poisson white noises [40]. They have studied the system with multiple peaks in the PDF [41], and a nonlinear impact oscillator [42,43]. In conjunction with the method of sub-space-slice, the EPC method has been used to predict the steady-state PDF of multi-degree-of-freedom nonlinear systems [44]. Paola and Sofi [45] generalized Er's EPC method with the help of a proper choice of the weighting functions, and suggested a simple and effective iterative procedure to improve the accuracy of the approximate solution.

In the current paper, an iterative method for the closed-form solution of the steady-state PDF of single degree of freedom nonlinear systems under external and parametric Gaussian white noise excitations is developed. The proposed method can obtain the exact steady-state PDF of the system, if it exists, starting from a weighting function in the form of Gaussian PDF or an approximate PDF obtained with the method of stochastic averaging, for example.

The rest of the paper is outlined as follows. Section 2 outlines the steps of the proposed methods. It starts with the reduced FPK equation, then applying the method of weighted residue to determine the undetermined coefficients of the assumed solution in terms of the exponential function of polynomials in the state variables plus a logarithmic term in Section 2.1. Section 2.3 introduces an iterative procedure to improve the accuracy of the solutions obtained with the method of weighted residue. Section 2.4 presents a progressively iterative procedure for strongly nonlinear systems. In Section 3, three examples with known exact solutions of the reduced FPK equation are studied to demonstrate the convergence of the iterative procedure. The accuracy of the approximate PDF is measured against the exact solution or the Monte Carlo simulations. Section 4 concludes the paper.

2. The proposed method

The proposed method to construct the analytical solution of the reduced FPK equation consists of three steps. We apply the method of weighted residue for determining the unknown parameters in the assumed solution, and propose an iterative procedure to improve the accuracy of the solution obtained with the method of weighted residue. As an extension of the method, we further propose a progressively iterative procedure in the nonlinear parameter space to obtain the PDFs for highly nonlinear stochastic systems.

2.1. The reduced FPK equation

Consider a Stratonovich stochastic differentiation equation for a single degree of freedom nonlinear oscillator under external and parametric Gaussian white noise excitations in the state space form as

$$\frac{dX}{dt} = Y, \qquad (1)$$
$$\frac{dY}{dt} = -g(X,Y) + \sum_{i=1}^{l} [h_i(X,Y)]W_i(t),$$

where $g(X, Y)$ includes the damping and restoring forces, $h_i(X, Y)$ are linear or nonlinear functions, $W_i(t)$ denote independent Gaussian white noises with intensities $2D_{ij}$ $(i, j = 1, 2, \dots, l)$.

The reduced FPK equation of the system reads,

$$\mathcal{L}[p] \equiv -\frac{\partial}{\partial x}[pm_1] + \frac{\partial}{\partial y}[pm_2] + \frac{\partial^2}{\partial y^2}[pb_{22}] = 0, \qquad (2)$$