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# Finding zeros of nonlinear functions using the hybrid parallel cell mapping method

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ABSTRACT

Analysis of nonlinear dynamical systems including finding equilibrium states and stability boundaries often leads to a problem of finding zeros of vector functions. However, finding all the zeros of a set of vector functions in the domain of interest is quite a challenging task. This paper proposes a zero finding algorithm that combines the cell mapping methods and the subdivision techniques. Both the simple cell mapping (SCM) and generalized cell mapping (GCM) methods are used to identify a covering set of zeros. The subdivision technique is applied to enhance the solution resolution. The parallel implementation of the proposed method is discussed extensively. Several examples are presented to demonstrate the application and effectiveness of the proposed method. We then extend the study of finding zeros to the problem of finding stability boundaries of potential fields. Examples of two and three dimensional potential fields are studied. In addition to the effectiveness in finding the stability boundaries, the proposed method can handle several millions of cells in just a few seconds with the help of parallel computing in graphics processing units (GPUs).



## 1. Introduction

Finding zeros of multi-variable nonlinear functions is a common problem existing in many scientific and engineering fields. In the area of dynamics, finding equilibrium states of nonlinear systems, bifurcation and stability analysis of the system all lead to zero finding of nonlinear functions. In control systems, the stability region in the parameter space can also be transformed to a zero finding problem. This paper presents an algorithm using the simple cell mapping and generalized cell mapping that can find zeros of multi-variable nonlinear functions in an efficient manner.

Since analytical solutions for zeros of nonlinear functions are in general difficult to obtain, there have been many studies of numerical methods for zero finding. The classical Newton's method with gradient information has been successfully applied to various problems [1]. A number of novel variations of Newton's method are popular choices for many applications [2,3]. Other algorithms are focused on the non-smooth or complex functions where the derivatives are not available [4].

Other studies have been carried out to find global solutions of the zero finding problem in certain domain by treating either gradient based or gradient free search algorithms as point-mappings in the parameter space, and studying the long term evolutionary of the point-mappings. The homotopy continuation method [5], cell mapping [2] and set-oriented method [6,7] have

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been applied by many scholars to search global solutions. The homotopy continuation method is performed in the continuous parameter space while the latter two methods are performed in the discretized parameter space. For problems with moderate to high dimensions, the point-wise methods become less feasible due to the increased need of computational efforts. The set-oriented method and its predecessor, the cell mapping method, are more computationally effective.

Either gradient based or gradient free point-to-point iterative search algorithm for finding zeros forms a point-mapping dynamical system in the parameter space. Hence, finding function zeros can be equivalently treated as finding global invariant sets of the point-mapping dynamical system. Both the cell mapping and set-oriented methods were originally developed for finding global invariant sets of nonlinear dynamical systems. The cell mapping technique was originally developed by Hsu in the 1980s [8] for global analysis of strongly nonlinear dynamical systems. One prominent feature of the cell mapping is that it stores only short term trajectories of the dynamical system in the discrete cell state space. The cell mapping is constructed by converting point-to-point mapping to cell-to-cell mapping. Each cell is designated by its central point and is assigned with an $n$-tuple integer coordinate where $n$ is the dimension of the cell state space. An unravelling algorithm has been developed to extract global information from the cell mapping [8]. Two types of cell mappings, namely the simple cell mapping (SCM) and generalized cell mapping (GCM), have been developed. The simple cell mapping accepts only one image cell of a given cell, while the generalized cell mapping allows multiple image cells. The cell mapping method has been successfully applied in many fields (See references in [9]).

A further development of the cell mapping with an addition of subdivision technique, known as the set-oriented method, has been introduced by Dellnitz and Hohmann [10]. The set-oriented method with subdivision integrates an adaptive refinement technique. The advantage of the set-oriented method is that it can adaptively enhance the solution resolution with relatively lower computational effort by focusing on the invariant set only. This is known as the subdivision and selection procedure [11]. There have been a number of studies with the set-oriented method on searching invariant set and unstable manifolds. An adaptive subdivision algorithm was developed in [11] that allows the coexistence of multiple cell sizes to cover the invariant set. A study of global attractors of non-smooth mechanical system was carried out by Neumann *et al.* using the standard set-oriented method [12]. The set-oriented method is also a powerful tool in designing optimal controls [13,14], especially for multi-objective optimal controls [15,16]. Furthermore, the set-oriented method has shown great performance with the capability of locating all solutions in both real and complex domains [6,7].

This paper presents an algorithm of finding zeros of nonlinear functions in real domain. A SCM-GCM hybrid method in conjunction with the subdivision technique is developed to find zeros of nonlinear functions. For zero finding problems in high dimensional space, major components of the SCM-GCM hybrid method are parallelized. The parallel computing is carried out in a graphics processing unit (GPU) based upon the CUDA architecture. As a variant of zero finding studies, a searching algorithm for stability boundaries of nonlinear dynamical systems is proposed. The boundary searching algorithm makes use of the parallel simple cell mapping.

The reminder of this paper is organized as follows. Section 2 states the zero finding problem. Section 3 reviews the cell mapping methods, namely, the simple cell mapping and generalized cell mapping. Section 4 presents the detail of the algorithm and discusses the parallelization of the SCM-GCM hybrid method. Section 5 validates the method with several test examples. An analysis is carried out to study the convergence of the solutions in the cell space. The algorithm for searching stability boundaries with the parallel simple cell mapping is presented and validated in Section 6. We close the paper with concluding remarks in Section 7.

## 2. Zero finding algorithm as point mapping

Consider the problem of finding zeros in the following

$$\mathbf{f}(\mathbf{x}) = 0, \quad \mathbf{f} : \mathbf{R}^m \to \mathbf{R}^n, \ \mathbf{x} \in \mathbf{U} \subset \mathbf{R}^m, \tag{1}$$

where $\mathbf{U}$ is in a bounded region in $\mathbf{R}^m$. As an example, we consider the damped Newton's method for local search of zeros,

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \gamma \mathbf{J}^{\dagger}(\mathbf{x}_k)\mathbf{f}(\mathbf{x}_k) \equiv \mathbf{F}(\mathbf{x}_k), \tag{2}$$

where $\mathbf{J}\dagger$ denotes the pseudo inverse of the Jacobian matrix $\mathbf{J}$ of the function $\mathbf{f}(\mathbf{x}_k)$ at $\mathbf{x}_k$, and $\gamma$ is a step size. Note that the algorithm allows both under-determined $(m > n)$ and over-determined $(m < n)$ cases. When no explicit expression of the function $\mathbf{f}(\mathbf{x})$ is available, the Jacobian matrix $\mathbf{J}$ must be numerically computed with the methods such as the finite difference.

We are not interested in the case when the system has one simple zero in the region $\mathbf{U}$. Our goal is to find all the zeros of the vector-valued function $\mathbf{f}(\mathbf{x})$ for $\mathbf{x} \in \mathbf{U}$. In particular, we would like to find the zeros that form the set with certain global structure. Such zero solutions would require prohibitively extensive point-wise searches. However, the cell mapping technique is extremely attractive for this kind of problems.

Eq. (2) defines a point-to-point mapping $\mathbf{F}$: $\mathbf{R}^m \to \mathbf{R}^m$. We introduce the concept of the global attractor of the mapping $\mathbf{F}$. Define a set $A$ such that

$$\mathbf{F}(A) = A, \quad A \in \mathbf{U}. \tag{3}$$

Set $A$ is called the global attractor in $\mathbf{U}$ if $\cup_{i=1}^{\infty} \mathbf{F}^{-i}(A_\varepsilon) = \mathbf{U} \subset \mathbf{R}^m$, where $A_\varepsilon$ is a set covering $A$ and its $\varepsilon$-neighborhood. By definition, for any $\mathbf{z} \in A$, $\mathbf{F}^i(\mathbf{z}) \in A$ for $\forall i \in N^+$. $N^+$ denotes the set of the positive integers. Moreover, $\cup_{i=1}^{\infty} \mathbf{F}^i(B) \subseteq A$ holds if $B \subseteq A$.