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On the convergence of a new reliable algorithm for solving multi-order fractional differential equations

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ABSTRACT

In this paper, we will introduce the reconstruction of variational iteration method (RVIM) to solve multi-order fractional differential equations (M-FDEs), which include linear and non-linear ones. We will easily obtain approximate analytical solutions of M-FDEs by means of the RVIM based on the properties of fractional calculus. Moreover, the convergence of proposed method will be shown. Our scheme has been constructed for the fully general set of M-FDEs without any special assumptions, and is easy to implement numerically. Therefore, our method is more practical and helpful for solving a broad class of M-FDEs. Numerical results are carried out to confirm the accuracy and efficiency of proposed method. Several numerical examples are presented in the format of table and graphs to make comparison with the results that previously obtained by some other well known methods.

1. Introduction

The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order. Fractional calculus generalizes the notion of derivative for those cases in which the differentiation order is not natural number; in other words, it gives meaning to the expression $\frac{d^\alpha}{dt^\alpha} f(t)$ in those cases where α is a fraction or an irrational number. This generalization may be performed in several ways, leading to several slightly different definitions that do not always reach exactly at the same results. The Riemann–Liouville operator and the Caputo operator commonly are used by authors. Over the last decades, the use of fractional order derivatives has become more and more attractive in the broad field of engineering to describe different kinds of problems. It is well known that the integer order differential operators are local, while the most important profit of using fractional differential equations (FDEs) is their nonlocal property. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. Therefore, the memory effect of these derivatives is one of the main reasons to use them in various applications. Since the fractional calculus is a powerful tool to describe physical systems that have a long-term memory, new possibilities appear in mathematics and theoretical physics. Therefore, FDEs have gained popularity for describing various phenomena, for instance in visco elasticity [1], colored noise [2], signal processing [2], control theory [3], anomalous diffusion [4] notably in chaotic systems [5] and in phase transitions [6]. In study of FDEs, one should note that finding an analytical or approximate solution is a challenging problem. Therefore, accurate methods for finding the solutions of FDEs are yet under investigation. Several numerical methods for solving FDEs exist in the literature for example, Laplace transform method (Podlubny [6]), Fourier transform method (Kemple and Beyer [7]), Adomian's decomposition method (Daftardar-Gejji and Jafari [8,9]), Homotopy analysis method (Liao [10], Momani and Odibat [11], Yildirim [12], Kumar and Singh [13], Atangana and Secer

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[14]), Sumudu transform (Singh et al. [15], Sushila et al. [16], Atangana and Baleanu [17]), fractional Adams–Moulton method (Galeone and Garrappa [18]) and so on.

Multi-order fractional differential equations (M-FDEs) have been used to model various types of visco-elastic damping (see [6,19,20]). Some numerical methods have been investigated for solving M-FDEs such as operational matrix [21], Galerkin finite element method [22], predictor-corrector method [23], spectral collocation method [24] and Adams method [25]. Moreover, very few algorithms for the analytical solution of M-FDEs have been suggested [9,26,27] . And many of these methods are essentially used for particular types of these equations, often just linear ones or even smaller classes. It should be noted that, most of these methods cannot be generalized to nonlinear cases. The variational iteration method (VIM) that was first introduced by He [28] as a modification of the general Lagrange multiplier method [29] has been successfully applied to many ordinary and partial differential equations [30,31].

This encouraged researchers to extend this method for FDEs. So this extension is done and named as fractional variational iteration method (FVIM). Up to now, this method has been applied for solving different kinds of FDEs [32,33]. Sweilam et al. have used FVIM for solving M-FDEs [34]. They converted the equation into a system of FDEs and then applied FVIM to the resulting system. Not only these authors, but also some other authors did this strategy in order to solve M-FDEs. The shortcoming of this strategy is that, if the order of the M-FDE is very large, then the fractional differential system has many equations, so it is difficult to find analytical solution for this system.

In 2015, we proposed a new alternative approach based on the variational iteration formulations and Laplace transform for solving fractional partial differential equations which was called the reconstruction of variational iteration method (RVIM) [35].

In the present work, we extend the RVIM for solving multi-order fractional differential equations, which include linear and nonlinear ones. We can easily obtain approximate analytical solutions of M-FDEs by means of the RVIM and a few simple transformations which are based on the properties of the fractional calculus. Moreover, the convergence of proposed method will be studied. Our aim is to provide an accurate scheme that is robust, reliable, and reasonably inexpensive in terms of both set-up costs and the time taken to execute. Our scheme has been constructed for the fully general set of M-FDEs without any special assumptions, and is easy to implement numerically. It is worth mentioning that the proposed method is capable of reducing the volume of the computational work as compared to some other classical methods. The outline of this paper is as follows. Section 2 contains preliminaries. In Section 3, the RVIM is developed to solve the M-FDEs. Section 4 is devoted to the convergence of proposed method. In Section 5, extensive numerical experiments are presented to illustrate the accuracy and efficiency of our method. Finally, discussion and conclusion are summarized in Section 6.

2. Preliminaries

In this section, we recall the basic definitions and operational properties of fractional integral and derivative. Many definitions of fractional calculus have been proposed in the past two centuries. These definitions include Riemann–Liouville, Reize, Caputo and Grnwald–Letnikov fractional operators. The two most commonly used definitions are the Riemann–Liouville operator and the Caputo operator. In this part, we enlist some definitions and properties of the fractional calculus.

Definition 2.1. [36] A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p(> \mu)$, such that $f(t) = t^p v(t)$, where $v(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m if $f^{(m)} \in C_\mu$,$m \in N$.

Definition 2.2. [36] The Riemann–Liouville fractional integral operator of order $\alpha \geq 0$, of a function $f(t) \in C_\mu$, $\mu \geq -1$ is defined as:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha > 0, t > 0,$$

such that $I^0 f(t) = f(t)$. Note that Γ is the gamma function.

For the Riemann–Liouville fractional integral we have the following properties:

$$I^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)}(t-a)^{\beta+\alpha},$$
$$I^\alpha I^\beta f(t) = I^\beta I^\alpha f(t) = I^{\alpha+\beta} f(t).$$

Definition 2.3. [36] The Riemann–Liouville fractional derivative of f, $f(t) \in C_\mu$ of order $\alpha \geq 0$ is defined as:

$$D^\alpha_{R-L} f(t) = D^n I^{n-\alpha} f(t).$$

Definition 2.4. [37] For a continuous function $f(t)$, the Caputo fractional derivative of order α where $n-1 < \alpha < n$, is defined as

$$D^\alpha_c f(t) = I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} D^n f(\tau) d\tau.$$

It has the following property:

$$I^\alpha D^\alpha_c f(t) = f(t) - \sum_{k=0}^{n} f^{(k)}(0^+)\frac{x^k}{k!}.$$