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Symmetries, solutions and conservation laws of a class of nonlinear dispersive wave equations

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ABSTRACT

In this paper we consider a damped externally excited Korteweg-de Vries (KdV) equation with a forcing term. We derive the classical Lie symmetries admitted by the equation. We then find that the damped externally excited KdV equation has some exact solutions which are periodic waves and solitary waves. These solutions are derived from the solutions of a simple nonlinear ordinary differential equation. By using a general theorem on conservation laws and the multiplier method, we construct some conservation laws for some of these partial differential equations.

1. Introduction

In [1], an analytical model of Tsunami generation by sub-marine landslides was proposed. The model is given by

$$u_t + auu_x + bu_{xxx} + cu_x = -\frac{c}{2}z_x(x,t) \tag{1.1}$$

with $a = 3c/2h$, $b = ch^2/6$. Here $a \neq 0$, $b \neq 0$, $u = u(x,t)$ refers to the elevation of the free water surface, $z = z(x,t)$ represents the solid bottom, h is assumed to be the constant mean water depth and $c = (gh)^{1/2}$ is the long wave speed with g being the gravity acceleration.

When the right-hand side term of Eq. (1.1), viz., $\frac{c}{2}z_x(x,t)$, called the forcing term, equals zero, Eq. (1.1) is reduced to the classical KdV equation. Therefore, Eq. (1.1) is called a KdV equation with forcing term or the forced equation. In recent years, several explicit asymptotic derivations for this generic model Eq. (1.1) have been studied in [2,3]. We should note that in the absence of the forcing term $\frac{c}{2}z_x(x,t)$, the classical KdV equation is completely integrable [4–7] while the KdV equation with a forcing term is not known to be integrable.

In [8], analytic solutions were found for the forced KdV Eq. (1.1) using the Hirota's bilinear method for the special case when

$$z = -\frac{2}{c}xf(t) + \text{constant}$$

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with $f(t)$ being an arbitrary smooth function of t. In [9] for the forced KdV equation, periodic and solitary waves solutions were found by using the concept of weak self-adjointness and some conservation laws were derived.

In a recent paper, Eloe and Usman [10] studied the damped externally excited KdV equation

$$u_t + 2buu_x + au_{xxx} - cu_{xx} - du = \eta \cos (kx + \lambda t), \tag{1.2}$$

where η represents the amplitude of the forcing term, λ and k represents the velocity and wave constants. Here c and d are nonnegative constants that are proportional to the strength of the damping effect. The asymptotic perturbation method was used to analyze the stability of solutions.

Due to the interest in the damped externally excited KdV equation, in this paper we study the generalization of the damped externally excited KdV equation studied in [10], namely

$$u_t + 2buu_x + au_{xxx} - cu_{xx} - du = f(x, t) \tag{1.3}$$

from the point of view of Lie symmetry analysis. Here c and d are nonnegative constants that are proportional to the strength of the damping effect. This damping effect has not been considered in previous papers [8,9].

We obtain the classical symmetries admitted by (1.3) for an arbitrary function f and the functional forms of f for which Eq. (1.3) admits extra classical symmetries. Then we use the transformation groups to reduce the equations to ODEs. In order to derive further reductions of these equations and search for more traveling wave solutions of (1.3), we employ an auxiliary simple equation method. While looking for traveling wave solutions we find that for some fixed forcing terms the reduced ordinary differential equation (ODE) has some exact solutions that can be expressed in terms of the Jacobi elliptic functions and consequently it has some exact solutions that can be expressed in terms of trigonometric and hyperbolic functions. Hence, (1.3) has some solutions which are periodic and solitary waves. Among these we find curved solitons solutions. We then determine, for (1.3), the class of equations which are nonlinearly self-adjoint and derive, by using the notation and techniques of [11] and the direct method of the multipliers [12], some nontrivial conservation laws of Eq. (1.3).

2. Classical symmetries

In this section we perform Lie symmetry analysis for Eq. (1.3). Let us consider a one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x, t, u) + \mathcal{O}(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x, t, u) + \mathcal{O}(\varepsilon^2), \\ u^* &= u + \varepsilon\phi(x, t, u) + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{2.1}$$

where ε is the group parameter. Then one requires that this transformation leaves invariant the set of solutions of the Eq. (1.3). This yields an overdetermined, linear system of partial differential equations for the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$ and $\phi(x, t, u)$. The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \phi(x, t, u)\frac{\partial}{\partial u}. \tag{2.2}$$

Having determined the infinitesimals, the symmetry variables are found by solving the invariant surface condition

$$\Phi \equiv \xi\frac{\partial u}{\partial x} + \tau\frac{\partial u}{\partial t} - \phi = 0. \tag{2.3}$$

Solving this system we find that

$$\xi = \delta(t), \qquad \tau = k_1, \qquad \phi = \frac{\delta_t}{2b},$$

where $\delta(t)$, $f(x, t)$ and k_1 must satisfy the condition

$$2\,b\,f_t\,k_1 + 2\,b\,\delta\,f_x - \delta_{tt} + d\,\delta_t = 0. \tag{2.4}$$

We can state that if $f = f(x, t)$ does not satisfy (2.4), then Eq. (1.3) does not admit any classical symmetry. This means that, for any $f = f(x, t)$ satisfying (2.4), Eq. (1.3) admits the classical generator

$$\mathbf{v} = \delta\frac{\partial}{\partial x} + k_1\frac{\partial}{\partial t} + \frac{\delta_t}{2b}\frac{\partial}{\partial u}. \tag{2.5}$$

We can distinguish the following cases:

Case 1. If $f_x(x, t) = 0$, then for any arbitrary $f = f(t)$ Eq. (1.3) admits the classical generator (2.5) where $\delta(t)$ is given by

$$\delta(t) = \frac{2\,b\,k_1\,e^{d\,t}}{d}\int e^{-d\,t}\,f_t\,dt - \frac{2\,b\,k_1}{d}\,f + k_2\,e^{d\,t} + k_3. \tag{2.6}$$