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Research paper

Equivalence transformations and conservation laws for a generalized variable-coefficient Gardner equation

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ABSTRACT

In this paper we study the generalized variable-coefficient Gardner equations of the form $u_t + A(t)u^n\, u_x + C(t)\, u^{2n}u_x + B(t)\, u_{xxx} + Q(t)\, u = 0$. This class broadens out many other equations previously considered: Johnpillai and Khalique (2010), Molati and Ramollo (2012) and Vaneeva et al. (2015). The use of the equivalence group of this class allows us to perform an exhaustive study and a simple and clear formulation of the results. Some conservation laws are derived for the nonlinearly self-adjoint equations by using a general theorem on conservation laws. We also construct conservation laws by applying the multipliers method.

1. Introduction

Recent developments in the field of partial differential equations (PDEs) have led researchers to focus their efforts on the study of PDEs with variable coefficients, particularly in nonlinear equations with variable coefficients. These equations describe many nonlinear phenomena more realistically than their constant-coefficient counterparts. Nonlinear evolution equations play an important role in the field of nonlinear dynamics. Among them, we draw special attention to the Korteweg–de Vries (KdV) equation and its generalizations. The KdV equation has been used to model nonlinear problems with great physical interest in mathematical physics, nonlinear dynamics and plasma physics. The issue lies in the fact that the KdV equation is quite a simple equation to analyze these phenomena. Thus, generalizations of the KdV equation which involve more than one nonlinear term must be considered. For instance, the Gardner equation, also known as combined KdV–mKdV equation, is a useful model for the description of wave phenomena in plasma and solid state and internal solitary waves in shallow waters.

Recently there has been much interest in the Gardner equation. In [11] Johnpillai and Khalique considered the generalized KdV equation with time dependent coefficients given by

$$u_t + u\, u_x + B(t)\, u_{xxx} + Q(t)\, u = 0, \tag{1}$$

where $B(t)$ and $Q(t)$ are arbitrary smooth functions of t. Here, the third term represents the dispersion term while the fourth term is the linear damping. The time dependent coefficients of dispersion and damping are, respectively, $B(t)$ and $Q(t)$. The authors obtained the optimal system of one-dimensional subalgebras of the Lie symmetry algebras of class (1). In [12] some

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conservation laws for class (1) were constructed for some special forms of $B(t)$ and $Q(t)$. Molati and Ramollo obtained the Lie symmetries of the variable-coefficient Gardner equation given by

$$u_t + A(t)\,u\,u_x + C(t)\,u^2 u_x + B(t)\,u_{xxx} = 0, \tag{2}$$

where $A(t)$, $B(t)$ and $C(t)$ are smooth functions of t verifying $B \cdot C \neq 0$ [15]. Vaneeva et al. [20] enhanced the classification of Lie symmetries obtained in [15] by using the generalized extended equivalence group.

In this paper, we broaden out the previous results by considering the generalized variable-coefficient Gardner equation with nonlinear terms of any order

$$u_t + A(t)\,u^n\,u_x + C(t)\,u^{2n} u_x + B(t)\,u_{xxx} + Q(t)\,u = 0, \tag{3}$$

where n is a positive constant, $A(t)$, $B(t) \neq 0$, $C(t) \neq 0$ and $Q(t) \neq 0$ are arbitrary smooth functions of t.

The analysis of nonlinear equations involving arbitrary functions is a rather difficult task. Thus, one could expect that there was a transformation which maps Eq. (3) into another equation from the same class with a smaller number of arbitrary elements. This transformation can be obtained by using the gauging of arbitrary elements by equivalence transformations.

By definition, an equivalence transformation is a non-degenerate change of the dependent and independent variables with the property that it maps every equation of class (3) into an equation of the same class, i.e., into an equation preserving the same differential structure but with different arbitrary functions. The main advantage of using equivalence transformations is that instead of considering individual equations, one can develop an analysis for complete equivalent classes. Therefore, the use of equivalence transformations provides a powerful tool for studying PDEs with variable coefficients.

The symmetry group of a PDE is the largest group of transformations acting on the space of independent and dependent variables which transforms solutions of the equation into other solutions. One of the most powerful methods available to analyze PDEs is the method of Lie symmetry groups. Symmetry groups have several well-known applications. For instance, they can be used to obtain exact solutions [4,18,21] or to construct conservation laws [5–9,19].

Given a PDE, a conservation law is a space-time divergence expression which vanishes on all solutions of the PDE. Although this concept has its origin in physics, it has broad application in many other areas of science. In mathematics, they can be used in numerical methods and mathematical analysis to investigate the existence, uniqueness and stability of solutions of PDEs. Furthermore, the existence of a large number of conservation laws of a PDE is a strong indication of its integrability.

This work is organized as follows. In Section 2, we obtain the continuous equivalence group of Eq. (3). Next, in Section 3 we get the Lie symmetries of the reduced equation which has been obtained using the gauging of arbitrary functions by equivalence transformations. In Sections 4 and 5 we use the concept of adjoint equation and we determine the subclasses of the equation which are nonlinearly self-adjoint. In Section 6, we obtain conservation laws by using a general theorem proved by Ibragimov [13] and a direct method proposed by Anco and Bluman [1,2]. The conclusions are presented in Section 7.

2. Equivalence transformations

In this section we determine the equivalence transformations of class (3). These transformations allow us to reduce class (3) to a subclass with a smaller number of arbitrary elements. An equivalence transformation of class (3) is a nondegenerate point transformation, (t, x, u) to $(\tilde{t}, \tilde{x}, \tilde{u})$ in the augmented space (t, x, u, A, B, C, Q, n) which transforms any equation of class (3) into an equation of the same class but with different arbitrary elements, $\tilde{A}(\tilde{t})$, $\tilde{B}(\tilde{t})$, $\tilde{C}(\tilde{t})$, $\tilde{Q}(\tilde{t})$ and $\tilde{n}$ from the original ones. We apply Lie's infinitesimal criterion [17] to obtain the equivalence transformations of class (3). However, in the case of the infinitesimal equivalence generator, we require not only the invariance of class (3) but also the invariance of the auxiliary system

$$A_x = A_u = B_x = B_u = C_x = C_u = Q_x = Q_u = n_t = n_x = n_u = 0. \tag{4}$$

We consider the one-parameter group of equivalence transformations in (t, x, u, A, B, C, Q, n) given by

$$\begin{aligned}
\tilde{t} &= t + \epsilon\,\tau(t,x,u) + O(\epsilon^2),\\
\tilde{x} &= x + \epsilon\,\xi(t,x,u) + O(\epsilon^2),\\
\tilde{u} &= u + \epsilon\,\eta(t,x,u) + O(\epsilon^2),\\
\tilde{A} &= A + \epsilon\,\omega^1(t,x,u,A,B,C,Q,n) + O(\epsilon^2),\\
\tilde{B} &= B + \epsilon\,\omega^2(t,x,u,A,B,C,Q,n) + O(\epsilon^2),\\
\tilde{C} &= C + \epsilon\,\omega^3(t,x,u,A,B,C,Q,n) + O(\epsilon^2),\\
\tilde{Q} &= Q + \epsilon\,\omega^4(t,x,u,A,B,C,Q,n) + O(\epsilon^2),\\
\tilde{n} &= n + \epsilon\,\omega^5(t,x,u,A,B,C,Q,n) + O(\epsilon^2),
\end{aligned} \tag{5}$$

where ϵ is the group parameter. In this case, the vector field takes the following form

$$Y = \tau\,\partial_t + \xi\,\partial_x + \eta\partial_u + \omega^1\partial_A + \omega^2\partial_B + \omega^3\partial_C + \omega^4\partial_Q + \omega^5\partial_n. \tag{6}$$