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Unpredictable points and chaos

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ABSTRACT

It is revealed that a special kind of Poisson stable point, which we call an unpredictable point, gives rise to the existence of chaos in the quasi-minimal set. The existing definitions of chaos are formulated in sets of motions. This is the first time in the literature that description of chaos is initiated from a single motion. The theoretical results are exemplified by means of the symbolic dynamics.

1. Introduction

The mathematical dynamics theory, which was founded by Poincaré [1] and significantly developed by the French genius and Birkhoff [2], was a source as well as the basis for the later discoveries and thorough investigations of complex dynamics [3–7]. The homoclinic chaos was discussed by Poincaré [8], and Lorenz [5] observed that a strange attractor contains a Poisson stable trajectory. Possibly, it was Hilmy [9,10] who gave the first definition of a quasi-minimal set (as the closure of the hull of a Poisson stable motion). In [10, p. 361] one can find a theorem by Hilmy, which states the existence of an uncountable set of Poisson stable trajectories in a quasi-minimal set. We modify the Poisson stable points to unpredictable points such that the quasi-minimal set is chaotic.

Let (X, d) be a metric space and $\mathbb{T}$ refer to either the set of real numbers or the set of integers. A mapping $f\colon \mathbb{T}\times X \to X$ is a flow on X [11] if:

(i) $f(0, p) = p$ for all $p \in X$;
(ii) $f(t, p)$ is continuous in the pair of variables t and p;
(iii) $f(t_1, f(t_2, p)) = f(t_1 + t_2, p)$ for all $t_1, t_2 \in \mathbb{T}$ and $p \in X$.

If a mapping $f\colon \mathbb{T}_+ \times X \to X$, where $\mathbb{T}_+$ is either the set of non-negative real numbers or the set of non-negative integers, satisfies (*i*), (*ii*) and (*iii*), then it is called a semi-flow on X [11].

Suppose that f is a flow on X. A point $p \in X$ is stable P^+ (positively Poisson stable) if for any neighborhood $\mathcal{U}$ of p and for any $H_1 > 0$ there exists $t \geq H_1$ such that $f(t, p) \in \mathcal{U}$. Similarly, $p \in X$ is stable P^- (negatively Poisson stable) if for any neighborhood $\mathcal{U}$ of p and for any $H_2 < 0$ there exists $t \leq H_2$ such that $f(t, p) \in \mathcal{U}$. A point $p \in X$ is called stable P (Poisson stable) if it is both stable P^+ and stable P^- [10].

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For a fixed $p \in X$, let us denote by Ω_p the closure of the trajectory $\mathcal{T}(p) = \{f(t, p) : t \in \mathbb{T}\}$, i.e., $\Omega_p = \overline{\mathcal{T}(p)}$. The set Ω_p is a quasi-minimal set if the point p is stable P and $\mathcal{T}(p)$ is contained in a compact subset of X [10]. We will also denote $\Omega_p^+ = \overline{\mathcal{T}^+(p)}$, where $\mathcal{T}^+(p) = \{f(t, p) : t \in \mathbb{T}_+\}$ is the positive semi-trajectory through p.

An essential result concerning quasi-minimal sets was given by Hilmy [9]. It was demonstrated that if the trajectory corresponding to a Poisson stable point p is contained in a compact subset of X and Ω_p is neither a rest point nor a cycle, then Ω_p contains an uncountable set of motions everywhere dense and Poisson stable. The following theorem can be proved by adapting the technique given in [9,10].

Theorem 1.1. *Suppose that $p \in X$ is stable P^+ and $\mathcal{T}^+(p)$ is contained in a compact subset of X. If Ω_p^+ is neither a rest point nor a cycle, then it contains an uncountable set of motions everywhere dense and stable P^+.*

2. Unpredictable points and trajectories

In this section, we will introduce unpredictable points and mention some properties of the corresponding motions. The results will be provided for semi-flows on X, but they are valid for flows as well. We will denote by $\mathbb{N}$ the set of natural numbers.

Definition 2.1. A point $p \in X$ and the trajectory through it are *unpredictable* if there exist a positive number ϵ_0 (the unpredictability constant) and sequences $\{t_n\}$ and $\{\tau_n\}$, both of which diverge to infinity, such that $\lim_{n\to\infty} f(t_n, p) = p$ and $d[f(t_n + \tau_n, p), f(\tau_n, p)] \geq \epsilon_0$ for each $n \in \mathbb{N}$.

An important point to discuss is the sensitivity or unpredictability. In the famous research studies [1,4,5,7,8,12], sensitivity was considered as a property of a system on a certain set of initial data since it compares the behavior of at least couples of solutions. The above definition allows to formulate unpredictability for a single trajectory. Indicating an unpredictable point p, one can make an error by taking a point $f(t_n, p)$. Then $d[f(\tau_n, f(t_n, p)), f(\tau_n, p)] \geq \epsilon_0$, and this is unpredictability for the motion. Thus, we say about the unpredictability of a single trajectory whereas the former definitions considered the property in a set of motions. In Section 3, it will be shown how to extend the unpredictability to a chaos.

The following assertion is valid.

Lemma 2.1. *If $p \in X$ is an unpredictable point, then $\mathcal{T}^+(p)$ is neither a rest point nor a cycle.*

Proof. Let the number ϵ_0 and the sequences $\{t_n\}$, $\{\tau_n\}$ be as in Definition 2.1. Assume that there exists a positive number ω such that $f(t+\omega, p) = f(t, p)$ for all $t \in \mathbb{T}_+$. According to the continuity of $f(t, p)$, there exists a positive number δ such that if $d[p, q] < \delta$ and $0 \leq t \leq \omega$, then $d[f(t, p), f(t, q)] < \epsilon_0$. Fix a natural number n such that $d[p_n, p] < \delta$, where $p_n = f(t_n, p)$. One can find an integer m and a number ω_0 satisfying $0 \leq \omega_0 < \omega$ such that $\tau_n = m\omega + \omega_0$. In this case, we have that

$$d[f(\tau_n, p_n), f(\tau_n, p)] = d[f(\omega_0, p_n), f(\omega_0, p)] < \epsilon_0.$$

But, this is a contradiction since

$$d[f(\tau_n, p_n), f(\tau_n, p)] = d[f(t_n + \tau_n, p), f(\tau_n, p)] \geq \epsilon_0.$$

Consequently, $\mathcal{T}^+(p)$ is neither a rest point nor a cycle. □

It is seen from the next lemma that the unpredictability can be transmitted by the flow.

Lemma 2.2. *If a point $p \in X$ is unpredictable, then every point of the trajectory $\mathcal{T}^+(p)$ is also unpredictable.*

Proof. Suppose that the number ϵ_0 and the sequences $\{t_n\}$, $\{\tau_n\}$ are as in Definition 2.1. Fix an arbitrary point $q \in \mathcal{T}^+(p)$ such that $q = f(\bar{t}, p)$ for some $\bar{t} \in \mathbb{T}_+$. One can verify that

$$\lim_{n\to\infty} f(t_n, q) = \lim_{n\to\infty} f(t_n + \bar{t}, p) = \lim_{n\to\infty} f(\bar{t}, f(t_n, p)) = f(\bar{t}, p) = q.$$

Now, take a natural number n_0 such that $\tau_n > \bar{t}$ for each $n \geq n_0$. If we denote $\zeta_n = \tau_n - \bar{t}$, then we have for $n \geq n_0$ that

$$\begin{aligned} d[f(t_n + \zeta_n, q), f(\zeta_n, q)] &= d[f(t_n + \zeta_n, f(\bar{t}, p)), f(\zeta_n, f(\bar{t}, p))] \\ &= d[f(t_n + \tau_n, p), f(\tau_n, p)] \\ &\geq \epsilon_0. \end{aligned}$$

Clearly, $\zeta_n \to \infty$ as $n \to \infty$. Consequently, the point q is unpredictable. □

Remark 2.1. It is worth noting that the unpredictability constant ϵ_0 is common for each point on an unpredictable trajectory.