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Short communication

Matrix fractional systems

J.A. Tenreiro Machado

Institute of Engineering of Polytechnic of Porto, Dept. of Electrical Engineering, Porto, Portugal

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ABSTRACT

This paper addresses the matrix representation of dynamical systems in the perspective of fractional calculus. Fractional elements and fractional systems are interpreted under the light of the classical Cole–Cole, Davidson–Cole, and Havriliak–Negami heuristic models. Numerical simulations for an electrical circuit enlighten the results for matrix based models and high fractional orders. The conclusions clarify the distinction between fractional elements and fractional systems.

1. Introduction

During the last decade Fractional Calculus (FC) witnessed a considerable progress [1–3], extending its scope from pure mathematics [4–7] up to real world applications [8–16]. The generalization of classical models and algorithms to their fractional counterpart represents a formidable volume of topics [17], but the fact is that many fundamental concepts remain untouched, or require further work towards building a comprehensive theory of fractional order systems.

In this study we address the fundamental concepts of fractional element (FE) and fractional system (FS). A FE is merely a limit case of a FS, that is, of a system constituted by a single element. However, this hides the interpretation of the general case of a system with multiple elements (of integer and fractional orders). While is commonly adopted the term FS, we shall verify that "fraction of the system" is more appropriate. To shed some light into this conceptual paradox we shall start by recalling the Debye, Cole–Cole, Davidson–Cole, and Havriliak–Negami (HN) heuristic models [18–26]. Usually, these models are adopted for low order scalar transfer functions. Identically, are recalled the concepts of FE that somehow 'interpolate' standard integer elements in electrical and mechanical systems [27–33]. However, it is possible not only to consider a wider variation of the fractional order, but also to develop a matrix representation. The matrix formulation of systems leads to the emergence of the concept of "fraction of the system". A prototype example is established by means of a simple electrical circuit represented in matrix form. Then, matrix fractions are analyzed in the Fourier domain for distinct orders and the results approximated numerically by means of HN based transfer functions. Alternatively, having in mind possible implementations, approximation circuits incorporating integer and fractional elements are also tested. The results demonstrate the conceptual difference between FE and FS in the multidimensional representation.

In this line of thought, the paper is organized as follows. Section 2 presents the fundamentals of the FS and FE. Section 3 develops numerical experiments for testing the proposed concepts and evaluates some approximations. Finally, Section 4 outlines the main conclusions.

E-mail address: jtm@isep.ipp.pt

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2. Fundamental concepts

In fractional order systems we have two transfer functions of particular interest, namely the Cole–Cole (or "explicit") and Davidson–Cole (or "implicit") models [19–24] given respectively by:

$$G(s) = \frac{K}{\left(\frac{s}{\omega_0}\right)^\alpha + 1}, \tag{1}$$

$$G(s) = \frac{K}{\left(\frac{s}{\omega_0} + 1\right)^\gamma}, \tag{2}$$

where $\alpha, \gamma \in \mathbb{R}, K, \omega_0 \in \mathbb{R}^+$ and s denotes the Laplace variable.

When $\alpha = 1$ and $\gamma = 1$ these expressions reduce to the Debye model [18]:

$$G(s) = \frac{K}{\frac{s}{\omega_0} + 1}. \tag{3}$$

On the other hand, the two can be embedded leading to the so-called HN transfer function [25,26]

$$G(s) = \frac{K}{\left[\left(\frac{s}{\omega_0}\right)^\alpha + 1\right]^\gamma}. \tag{4}$$

Denoting by $\mathcal{L}\{\cdot\}$ the Laplace operator and t time, are known the relationships:

$$\mathcal{L}\left\{\omega_0^\alpha t^{\alpha-1} E_{\alpha,\alpha}\left[-(\omega_0 t)^\alpha\right]\right\} = \frac{1}{\left(\frac{s}{\omega_0}\right)^\alpha + 1}, \tag{5a}$$

$$\mathcal{L}\left\{\omega_0^\gamma e^{-\omega_0 t}\frac{t^{\gamma-1}}{\Gamma(\gamma)}\right\} = \frac{1}{\left(\frac{s}{\omega_0} + 1\right)^\gamma}, \tag{5b}$$

$$\mathcal{L}\left\{\omega_0^{\alpha\gamma} t^{\alpha\gamma-1} E_{\alpha,\alpha\gamma}^{\gamma}\left[-(\omega_0 t)^\alpha\right]\right\} = \frac{1}{\left[\left(\frac{s}{\omega_0}\right)^\alpha + 1\right]^\gamma}, \tag{5c}$$

where $\Gamma(z), z \in \mathbb{C}$, is the Gamma function, and $E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$ and $E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)}\sum_{k=0}^{\infty}\frac{\Gamma(\gamma+k)z^k}{k!\Gamma(\alpha k+\beta)}$ represent the two and three parameters Mittag–Leffler functions, respectively [34–39].

The heuristic expressions (1)–(4) were introduced several decades ago [40] and have been adopted in many distinct areas [41–48], but applications are often restricted to one-dimensional cases.

Considerable attention has also been paid to the development of FE [27–33] with expressions:

$$Z(s) = Bs^\beta, \tag{6a}$$

$$Z(s) = \frac{1}{Ds^\delta}, \tag{6b}$$

where $A, B, \beta, \delta \in \mathbb{R}$.

Some researchers, refer to such inductive and capacitive elements as "fractductor" and "fractance", respectively, and proposed representations such as those depicted in Fig. 1 [49], but no definitive names and symbols were formulated. It is also interesting to note that the FE are merely particular cases of memristors, memcapacitors and meminductors [50–54].

We shall address here the multidimensional case and, consequently, before proceeding we need to clarify concepts. In fact, expression (1) means that we are considering a system incorporating one FE (of order α), while (2) implies that we are modeling a FS (of order γ) that includes an integer element. We can say loosely that in (1) we have one "complete" (integer) system and a "fraction" of the element and, on the other hand, in (2) we have a "fraction of the system" that includes a "complete" (integer) element. Obviously, model (4) considers the γth fraction of the system that includes the αth fraction of the element. Fig. 2 represents the locus of the Debye, Cole–Cole, Davidson–Cole, and Havriliak–Negami models.

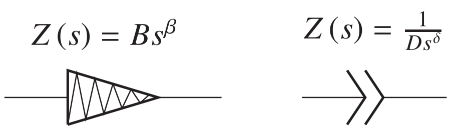

Fig. 1. FE symbols for inductive $Z(s) = Bs^\beta$ ("fractductor") and capacitive $Z(s) = \frac{1}{Ds^\delta}$ ("fractance").