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Stability of periodic traveling waves in the Aliev–Panfilov reaction–diffusion system

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ABSTRACT

We study the two-component Aliev–Panfilov reaction–diffusion system of cardiac excitation. It is known that the model exhibits spiral wave instability in two-dimensional spatial domains. In order to describe the spiral wave instability, it is important to understand periodic traveling wave instability resulting from the model. We determine the existence and stability of periodic traveling waves in the model. In addition, we calculate the stability boundary between stable and unstable periodic traveling waves in a two-dimensional parameter plane. It is observed that the periodic traveling waves express instability by a stability change of Eckhaus type. As a result, a stable wave bifurcates to an oscillating periodic traveling wave. We describe these phenomena by calculating the essential spectra of the waves. Furthermore, we study the stability of the waves as a function of the gaps between two nullclines. In two dimensions, we determine the spiral wave instability based on the stability boundary of the periodic traveling waves.

1. Introduction

A periodic traveling wave (PTW) solution is an important solution pattern in one dimension for many partial differential equations (PDEs) in nonlinear systems. Periodic traveling wave solutions (PTWs) were first studied in 1973 by Kopell and Howard, who used coupled reaction–diffusion equations for oscillatory systems [1]. They showed that waves with sufficiently low amplitude are always unstable whereas those with sufficiently high amplitude are always stable. PTWs have also been observed in biological [2–4], physical [5–7], chemical [8–10] and ecological systems [11–13]. These are often modeled using a two-component reaction–diffusion system of equations, e.g., [1,14–17]. The determination of PTW instability for excitable media is important for researchers owing to the complex spatiotemporal patterns exhibited by the instability. A powerful and standard method for studying the PTWs of PDEs is the method of continuation [18]. In this paper, we study the stability of PTWs numerically by using a two-component excitable reaction–diffusion system for cardiac cell dynamics [19]. The first and most widely used PDE model is the FitzHugh–Nagumo (FHN) model, which was developed by FitzHugh and Nagumo [14,15]. The FHN model is a simplified version of the Hodgkin and Huxley model [20], a four-variable (V, m, n, and h) ionic model that describes the fast-slow dynamics of an excitable system of a spiking neuron. Previous studies of PTWs focused on ionic models [21–23] and PDE models for excitable systems [17,24–26] and other systems [27–29]. The Aliev–Panfilov model, which is a modified version of the FHN model,

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is capable of improving the shape of the cardiac action potential [19]. The action potential of a cardiac cell (myocardium) has four different phases: resting membrane potential, rapid depolarization, plateau, and rapid repolarization. The cardiac action potential mainly differs from the neuronal action potential in terms of the duration of the action potential or plateau phase. In a nerve cell the action potential duration is approximately 1 ms. However, the cardiac action potential has a prolonged plateau phase lasting around 300 ms [30]. Previous studies [31–36] of the Aliev–Panfilov model focused on investigating only the spiral wave instability or the spiral chaos in two-dimensional numerical simulations. In the context of cardiac electrical activities, the spiral breakup phenomenon is known as ventricular fibrillation. Spiral waves are of course two-dimensional, but they become one-dimensional as one moves away from the center of the medium. A spiral wave approaches a periodic traveling wave solution at a sufficiently large distance from the center. Instability of the PTW will lead to spiral instability, and therefore, possible spiral breakup. To the best of our knowledge, no published work has investigated the stability of periodic traveling wave solutions in the Aliev–Panfilov model.

Thus, the aim of this paper is to study the stability of PTWs in the Aliev–Panfilov model, which is given by (1) and (2) later in the paper. We establish the existence and stability of the PTW solutions of the model in a two-dimensional parameter plane. This is done by the method of continuation via the continuation package WAVETRAIN [18]. It is observed that a PTW solution loses its stability in the model through a stability change of Eckhaus type. The family of PTWs with constant period crosses the stability boundary, and the waves bifurcate to an oscillating wave pattern. We also study the existence and stability of the PTWs as a function of the parameter b in (2), which is responsible for creating gaps between two nullclines at the right slow manifold. Moreover, we show stable and unstable spiral dynamics in model (2) in a one-parameter family of solutions.

The remainder of the paper is organized as follows. In Section 2, we describe the models and methods of computation. In Section 3, we present and discuss the results in one and two dimensions. Finally, we conclude the paper in Section 4.

2. Methods

2.1. Model

2.1.1. Original

The Aliev–Panfilov model [19] consists of two equations describing the fast and slow dynamics of an excitable medium, and it is given as follows:

$$\begin{aligned}\frac{\partial u}{\partial t} &= d_u \Delta u + ku(1-u)(u-a) - uv,\\ \frac{\partial v}{\partial t} &= d_v \Delta v + \epsilon(u,v)(ku(1+a-u) - v),\end{aligned} \tag{1}$$

where $\epsilon(u, v) = \epsilon_0 + \mu_1 v/(u + \mu_2)$ and the reaction terms $f(u, v) = ku(1-u)(u-a) - uv$ and $g(u, v) = \epsilon(u, v)(ku(1+a-u) - v)$ describe the local kinetics of variables u and v. The parameter ϵ, $0 < \epsilon \ll 1$, describes the ratio of the time scales of variables u and v. The fast activator variable u and the slow inhibitor variable v are known as the excitable and recovery variables, respectively. They are also referred to as the propagator and controller variables, respectively. The nullclines $(f(u, v) = 0, g(u, v) = 0)$ of (1) has been plotted in [19]. The u-nullcline $(u_t = 0)$ is N-shaped, as in the case of the standard FHN model [14,15]. However, the Aliev–Panfilov uses the term uv instead of v in the first equation of the FHN model. This improves the shape of the cardiac action potential. The v-nullcline $(v_t = 0)$ is not linear or monotone as in the FHN model, but it is quadratic. This type of nullcline geometry is more appropriate for the cardiac cell dynamics [19]. The parameter a is called the threshold. It lies between 0 and 1/2, i.e., $0 < a < 1/2$. For a small perturbation of u less than the threshold value, i.e., $u < a$, the system reverts to the rest state; otherwise (i.e., $u > a$), the system undergoes long excursions with fast-slow dynamics in the (u, v)-plane before reverting to the stationary or $(0, 0)$ state. Therefore, the system is an excitable system [37].

2.1.2. Modified

In this paper, we use the following variant of the Aliev–Panfilov model in order to control the distance between the two nullclines uniformly at the right branch (marked by "R" in Fig. 1(a)) of the uv-plane as reported in [17,38].

$$\begin{aligned}\frac{\partial u}{\partial t} &= d_u \Delta u + ku(1-u)(u-a) - uv,\\ \frac{\partial v}{\partial t} &= d_v \Delta v + \epsilon(u,v)(du(1+b-u) - v).\end{aligned} \tag{2}$$

In (2), there are two new parameters d and b instead of k and a, respectively, in the second equation of (1). As a result, we can slow down the solution profile on the slow manifold at the right branch by taking the value of b as small as required. The nullclines of (2) are plotted in Fig. 1(a). Both nullclines intersect each other at a point that is called the rest state of the excitable medium. The point corresponds to the (0, 0) state as shown in Fig. 1(a). Thus, there exists only one possible steady-state solution. The parameter b in (2) is responsible for the creation of gaps between the two nullclines at the right knee or branch of the uv-plane. It plays a crucial role in the model. Specifically, it controls the period of excitation of the medium and can have any value greater than 0, because the two nullclines intersect each other at the right knee when $b = 0$. The excitation period increases as b approaches 0 and decreases as b increases. This is crucial for controlling the velocity of the solution at the right slow manifold,