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Stochastic fuzzy differential equations of a nonincreasing type

Marek T. Malinowski*

Institute of Mathematics, Cracow University of Technology, ul. Warszawska 24, 31–155 Kraków, Poland

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ABSTRACT

Stochastic fuzzy differential equations constitute an apparatus in modeling dynamic systems operating in fuzzy environment and governed by stochastic noises. In this paper we introduce a new kind of such the equations. Namely, the stochastic fuzzy differential of nonincreasing type are considered. The fuzzy stochastic processes which are solutions to these equations have trajectories with nonincreasing fuzziness in their values. In our previous papers, as a first natural extension of crisp stochastic differential equations, stochastic fuzzy differential equations of nondecreasing type were studied. In this paper we show that under suitable conditions each of the equations has a unique solution which possesses property of continuous dependence on data of the equation. To prove existence of the solutions we use sequences of successive approximate solutions. An estimation of an error of the approximate solution is established as well. Some examples of equations are solved and their solutions are simulated to illustrate the theory of stochastic fuzzy differential equations. All the achieved results apply to stochastic set-valued differential equations.

1. Introduction

Many real-world phenomena evolving in time are affected by random noises. Therefore in their mathematical models the stochastic differential equations [1–3] are used. They incorporate the stochastic uncertainty that is present in some inputs and parameters of the model. However, the randomness is not the only source of uncertainty in dynamic systems. Many times a functional relationship between parameters cannot be determined precisely and some sets of possible values are determined instead of a one precise value. Multivaluedness can be treated as a symptom of uncertainty which is not of stochastic type. The multivalued mappings are crucial for certain areas in physics, including theory of defects in crystal, magnetic monopoles, vortices in superfluids and superconductors, gauge field structures [4,5]. They are also used for quantum stochastic differential inclusions [6–8].

The situation becomes different if the data of the dynamic system can derive from a human perception or judgment. This is very often encountered in the real world because the realistic data are uncertain and imprecise. For example, the size of a bacteria population watched under a microscope is difficult to be determined precisely. The individuals move permanently and change their positions. As a result of an observer's perception the size of population could be described by some linguistic variables as "around 150", "medium", "large". Such data are imprecise and it is known that they can be modeled successfully by using fuzzy sets [9–11] which apply greatly in industrial mathematics, especially in fuzzy controllers and fuzzy networks [12–22]. We can also mention some applications in fuzzy differential equations [23–35], in modeling transmission of worms in computer networks [36], a marine port selection [37] or zero-sum games [38].

* Tel.: +48 12 6282994.
E-mail address: malinowskimarek@poczta.fm

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A stochastic model for the evolution of the population growth can also be examined with fuzzy information available on the initial value of considered system. This motivates to study the stochastic fuzzy differential equations [39–46] which generalize the random fuzzy differential equations [29–32,35] and the deterministic fuzzy differential equations [23–28,33,34]. The latter are suggested as an appropriate tool of modeling phenomena in physics taking into account the Heisenberg uncertainty principle [47,48]. In terms of stochastic fuzzy differential equations, a population growth model was considered in [40,41].

In the integral form of the stochastic fuzzy differential equations there appear the fuzzy stochastic Lebesgue–Aumann integral, which is a fuzzy random variable, and the classical stochastic Itô integral embedded into fuzzy sets space [39–41,44,45]. Such a setting enables to obtain the solutions in the form of fuzzy stochastic processes with continuous trajectories. The papers [39–41,44,45] contain the studies on stochastic fuzzy differential equations in a form which is a natural extension of the classical single-valued (crisp) stochastic differential equations. Then the solutions have a property (see Theorem 3.8 [41]) that values of trajectories become fuzzier and fuzzier as time increases. This type of a propagation of the fuzziness is not desired always. For example, if an observed population dies out then some experts could claim that fuzziness contained in a linguistic description of the population's size should be decreasing in time, because a smaller number of the observed individuals could be counted more and more precisely.

In this paper we provide mainly a study on the new stochastic fuzzy differential equations of a nonincreasing type, i.e. with solutions having nonincreasing fuzziness in their values. Their form is peculiar. However, they reduce to the classical stochastic differential equations if the data are singleton-defined and single-valued. All the equation coefficients are considered to be random. Under some appropriate measurability conditions, the Lipschitz and a boundedness condition imposed on the equation coefficients, we prove the existence and uniqueness of solutions to the stochastic fuzzy differential equations of nonincreasing type. An additional condition of well-posedness of the sequence of approximate solutions is indispensable here. We estimate a distance of nth approximation and exact solution. The stochastic fuzzy differential equations of nonincreasing type make a complement to the stochastic fuzzy differential equations of nondecreasing type [41]. The new type of stochastic fuzzy differential equations is well fitted to model the dynamics subjected to random perturbations and with nonincreasing uncertainty generated by fuzziness.

The structure of the paper is as follows: in Section 2 we summarize the background material concerning measurable multifunctions, fuzzy sets, fuzzy random variables and fuzzy stochastic Lebesgue–Aumann integral. In Section 3 we introduce the stochastic fuzzy differential equations of nonincreasing type and prove that their solutions have nonincreasing fuzziness. Beside that we show existence and uniqueness of solution to the nonincreasing type equation and this is done under an appropriate system of conditions imposed on the equation coefficients. The continuous dependence of solution with respect to data of the nonincreasing type equation is shown. Also the parallel results for the stochastic fuzzy differential equations of nondecreasing type are stated. We remark that in the case when the uncertainty is generated by ordinary sets one can use a setting of the stochastic set-valued differential equations. Since they are particular cases of the stochastic fuzzy differential equations, all the results established in Section 3 can be repeated for the stochastic set-valued differential equations. Section 4 contains some computations leading to a closed explicit form of solutions to a stochastic fuzzy model of population growth and stochastic fuzzy model of nuclear decay. For visualizations, we present four possible expert's choices of modeling with nonincreasing type and nondecreasing type stochastic fuzzy equations. With some concrete data we simulate numerically trajectories of solutions to the considered problems.

2. Preliminaries

Let $\mathcal{K}(\mathbb{R}^d)$ be the family of all nonempty, compact and convex subsets of $\mathbb{R}^d$. In $\mathcal{K}(\mathbb{R}^d)$ we consider the Hausdorff metric d_H which is defined by (cf. [49])

$$d_H(A,B) := \max\left\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right\},$$

where $\|\cdot\|$ denotes a norm in $\mathbb{R}^d$. It is known (see [49]) that $\mathcal{K}(\mathbb{R}^d)$ is a complete and separable metric space with respect to d_H.

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space and $\mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbb{R}^d))$ denote the family of $\mathcal{A}$-measurable multifunctions with values in $\mathcal{K}(\mathbb{R}^d)$, i.e. the mappings $F : \Omega \to \mathcal{K}(\mathbb{R}^d)$ such that

$$\{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\} \in \mathcal{A} \quad \text{for every open set} \quad O \subset \mathbb{R}^d.$$

A multifunction $F \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbb{R}^d))$ is said to be L^p-integrally bounded, $p \geq 1$, if there exists $h \in L^p(\Omega, \mathcal{A}, P; \mathbb{R})$ such that $\|a\| \leq h(\omega)$ for any a and ω with $a \in F(\omega)$. It is known (see [50]) that F is L^p-integrally bounded iff $\omega \mapsto d_H(F(\omega), \{0\})$ is in $L^p(\Omega, \mathcal{A}, P; \mathbb{R})$, where $L^p(\Omega, \mathcal{A}, P; \mathbb{R})$ is a space of equivalence classes (with respect to the equality P-a.e.) of $\mathcal{A}$-measurable random variables $h : \Omega \to \mathbb{R}$ such that $\mathbb{E}|h|^p = \int_\Omega |h|^p dP < \infty$.

Let us denote

$$\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(\mathbb{R}^d)) := \{F \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbb{R}^d)) : F \ \text{ is } \ L^p\text{-integrally bounded}\}.$$

The multifunctions $F, G \in \mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(\mathbb{R}^d))$ are considered to be identical, if $F = G$ holds P-a.e.

A fuzzy set u in $\mathbb{R}^d$ (see [9]) is characterized by its membership function (denoted by u again) $u : \mathbb{R}^d \to [0, 1]$ and $u(x)$ (for each $x \in \mathbb{R}^d$) is interpreted as the degree of membership of x in the fuzzy set u. As the value $u(x)$ expresses "degree of membership of