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A hybrid algorithm for Caputo fractional differential equations

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ABSTRACT

This paper is concerned with the numerical solution of fractional initial value problems (FIVP) in sense of Caputo's definition for dynamical systems. Unlike for integer-order derivatives that have a single definition, there is more than one definition of non integer-order derivatives and the solution of an FIVP is definition-dependent. In this paper, the chief differences of the main definitions of fractional derivatives are revisited and a numerical algorithm to solve an FIVP for Caputo derivative is proposed. The main advantages of the algorithm are twofold: it can be initialized with integer-order derivatives, and it is faster than the corresponding standard algorithm. The performance of the proposed algorithm is illustrated with examples which suggest that it requires about half the computation time to achieve the same accuracy than the standard algorithm.

1. Introduction

In recent decades, fractional or non-integer calculus has been applied in a wide range of fields [1–3]. Unlike for integer-order systems, in fractional calculus there is more than one definition for a non-integer derivative and the final results depend on the definition used.

A common goal in this field is to find solutions to fractional-order differential equations using numerical integration as a means for approximating the solutions [4–6]. In such a setting, often long time series need to be obtained numerically in related problems such as the analysis of a strange attractors [7–10] or the computation of bifurcation diagrams [11–14]. However, the intrinsic non local property of non-integer calculus requires very long computing times to obtain numerical solutions of fractional integrals, derivatives or differential equations.

Because of the importance of numerical methods, this issue has been investigated by many authors. Lubich obtained approximations for a Riemann–Liouville integral by a discrete convolution quadratures [15]. Podlubny used the Grünwald–Letnikov definition to obtain numerical solutions for fractional derivatives and differential equations for Riemann–Liouville definition [3]. Charef et al. obtained a linear approximation for the fractional-order integrator in the frequency domain, using a specified error in decibels and a bandwidth to generate a continuous sequence of pole-zero pairs for the system with a single fractional power pole [16]. Diethelm et al. proposed a method for fractional differential equations in the sense of Caputo using the equivalence of the solution of a fractional initial value problem by an equivalent Volterra integral equation and approximate this integral by a

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trapezoidal quadrature [4]. Yan et al. have generalized the method proposed in [4] approximating a Volterra integral equation by a piecewise quadratic polynomial and obtained better approximations [17].

The aim of this paper is to propose a hybrid method that improves on the algorithms described by Diethelm et al. [4] and Podlubny [3] for computing numerical solutions of fractional differential equations for Caputo definition which can be initialized with integer-order derivatives. As a by-product a relation between a fractional initial value problem in the sense of Caputo with its Riemann–Liouville counterpart is given in this paper.

The reminder of the paper is organized as follows: in Section 2, the definitions used in this paper are revisited. The main contribution of the paper is presented in Section 3 where a hybrid method to solve a fractional differential equation in Caputo's sense is put forward and in Section 4 this method is compared with the method introduced in [4]. The main points of the paper are summarized in Section 5.

2. Background

2.1. Definitions

The most commonly used definitions of fractional calculus are Riemann–Liouville derivative, defined as

$$ {}^{RL}D_a^{\alpha} f(t) := \frac{d^n}{dt^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1} f(\tau)\,d\tau\right), \tag{1} $$

and Caputo derivative, defined as

$$ {}^{C}D_a^{\alpha} f(t) := \left(\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,d\tau\right), \tag{2} $$

where $t \geq a$, $\Gamma(\,\cdot\,)$ is the Eüler gamma function, $\alpha > 0$, $n \in \mathbb{Z}$ such that $n-1 < \alpha \leq n$ and $f^{(n)}(t) = \frac{d^n f}{dt^n}$, [3,18].

These derivatives are related if f is continuously differentiable at least of order $(n-1)$ at $t = a$ as follows

$$ {}^{C}D_a^{\alpha} f(t) = {}^{RL}D_a^{\alpha}(f(t) - T_{n-1}[f;a](t)), \tag{3} $$

where $T_{n-1}[f;a](t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k$, [18].

An important definition in fractional calculus used for numerical approximations of Riemann–Liouville derivatives is the Grünwald–Letnikov definition [3,19] which is defined as a limit of a fractional-order backward difference

$$ {}^{GL}D_a^{\alpha} f(t) := \lim_{N\to\infty} \frac{1}{(h_N)^{\alpha}} \sum_{k=0}^{N} c_k^{\alpha} f(t - kh_N), \tag{4} $$

for $\alpha > 0$; where $h_N = \dfrac{(t-a)}{N}$ and

$$ c_k^{\alpha} = (-1)^k \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)\Gamma(k+1)}. \tag{5} $$

The Riemann–Liouville and the Grünwald–Letnikov derivatives of a function f are equal for $t \in (a, b]$ if $f(t) \in C^{\lceil\alpha\rceil}[a, b]$, [3].

By the relation (3) of a function f, Caputo and Grünwald–Letnikov derivatives are related by

$$ {}^{C}D_a^{\alpha} f(t) = {}^{GL}D_a^{\alpha}(f(t) - T_{n-1}[f;a](t)). \tag{6} $$

if $f(t) \in C^n[a, b]$.

2.2. Fractional Caputo differential equations

A fractional initial value problem (FIVP) with respect to Caputos's definition is given by a fractional differential equation (FDE)

$$ {}^{C}D_0^{\alpha} y = f(t, y) \tag{7} $$

with the initial conditions $y_0, y_0^{(1)}, \ldots y_0^{(n-1)}$ given by

$$ y_0^{(i)} = \left.\frac{d^i y(t)}{dt^i}\right|_{t=0}, i = 0, \ldots, n-1. \tag{8} $$

If in Eq. (7) the function $f(t, y)$ is continuous in both variables and Lipschitz with respect the second the Caputo FIVP has a unique solution in the interval $(0, T]$, $T > 0$, [18]. For details of Riemann–Liouville FIVP and conditions for existence and uniqueness see [20].