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On the logistic equation subject to uncertainties in the environmental carrying capacity and initial population density

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ABSTRACT

It is recognized that handling uncertainty is essential to obtain more reliable results in modeling and computer simulation. This paper aims to discuss the logistic equation subject to uncertainties in two parameters: the environmental carrying capacity, K, and the initial population density, N_0. We first provide the closed-form results for the first probability density function of time-population density, $N(t)$, and its inflection point, t^*. We then use the Maximum Entropy Principle to determine both K and N_0 density functions, treating such parameters as independent random variables and considering fluctuations of their values for a situation that commonly occurs in practice. Finally, closed-form results for the density functions and statistical moments of $N(t)$, for a fixed $t > 0$, and of t^* are provided, considering the uniform distribution case. We carried out numerical experiments to validate the theoretical results and compared them against that obtained using Monte Carlo simulation.

1. Introduction

Uncertainties are inherent in ecological system modeling and must be taken into account to improve the predictability and accuracy of the estimates. In this context, many authors have introduced stochastic population models to investigate the effect of environmental variability and perturbation [1–16]. Here, we explore uncertainties present in the logistic model, which is commonly applied in the studies of human, plants and bacterial populations, as well as to evaluate economic growth.

The logistic model was introduced to describe population growth considering a self-limitation term that corrects the unlimited growth of the Malthusian model [17]. The classical logistic (or Verhulst's) equation is the nonlinear initial value problem (IVP)

$$\begin{aligned} &\frac{d}{dt}N(t) = a\,N(t)\left(1 - \frac{N(t)}{K}\right), \quad t > 0,\\ &N(0) = N_0, \end{aligned} \tag{1}$$

where $N(t)$ denotes the population density at time t, $a > 0$ is the intrinsic growth rate, $N_0 > 0$ is the population density at time $t = 0$ and $K > 0$ is the environmental carrying capacity. This latter parameter represents the absolute maximum number of individuals in the population and is determined based on the limiting resource available.

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The parameters in IVP (1) are commonly considered accurate. Nevertheless, to design meaningful and realistic models, it is crucial to take into account that they are imprecise due to both the implicit lack of information and the mistakes in the measurement process present in related problems. Several approaches are considered, including the use of random variables to represent such parameters [1–3,9,11,12]. In [12], for instance, the authors analyze the logistic equation (1) with noise fluctuations in the carrying capacity. A probabilistic description of the solution of a random SI-type epidemiological equation, a model derived from IVP (1), where uncertainty is considered in both the initial condition and the rate of decline in the proportion of susceptibles, is presented in [1].

The Monte Carlo method [18] can be useful in this context. Basically, it numerically solves appropriate equations for representative sets of realizations of random variables and then averages the computed functions. Besides being applied to a very broad range of both linear and nonlinear problems, the large computational cost and the difficult to generalize the results may be prohibitive.

The probability density function (pdf) is also well suited to mathematically model aleatory uncertainties, considering that it incorporates all the statistical information about the process. Since there is usually not enough data from the experiments, the pdf can be determined using the Maximum Entropy Principle (MEP). It solves an optimization problem to determine the most unbiased probability distribution conditioned upon the available information.

In this work, we first use the environmental carrying capacity pdf, f_K, and the initial population density pdf, f_{N_0} to compute closed-form results for the first pdf both of time-population density, $N(t)$, for a fixed $t > 0$, and of its inflection point, t^*.

To better represent the imprecise nature of the problem, we focus on the stochastic modeling of the logistic equation to introduce uncertainty into the K and N_0 parameters, treated here as independent random variables in IVP (1). Using the MEP, we determine the pdf of both parameters considering fluctuations of their values for a situation that commonly occurs in practice. The propagation of these uncertainties to $N(t)$, for a fixed $t > 0$, and t^* are also discussed. The exact pdf both of $N(t)$ and t^* are presented considering the uniform distribution case. In the numerical experiments, the obtained results are compared against that obtained using Monte Carlo simulation.

Finally, we also derive the exact statistical moments of $N(t)$ and the mean of t^* for the uniform distribution case.

When an imprecise parameter is described by some probabilistic distribution, we assume that it is possible to choose a representative value using specific statistical methods. Thus, the solution of IVP (1) is obtained after uncertainty handling. On the other hand, it is also possible to first solve IVP (1), and then handle the uncertainty by treating such solution as a random variable, for fixed t. In this paper, we carried out computational tests to compare a simplified (standard) version of (1) where K and N_0 are replaced by their mean, $\mathrm{E}[K]$ and $\mathrm{E}[N_0]$, respectively, with the solution of the mean of the random solution, $N(t)$, of IVP (1).

The organization of this article is as follows: Sections 2 and 3 present closed-form results for the first pdf both of $N(t)$, for a fixed $t > 0$, and t^*. In Sections 4–6, we use the MEP to estimate these pdf when introducing uncertainty into the K and N_0 parameters. Closed-form results for the pdf and the statistical moments of $N(t)$, for a fixed $t > 0$, and t^* are provided. Finally, in Section 7, numerical simulations illustrate the main results.

2. Computation of the pdf of $N(t)$, for a fixed $t > 0$

In this section, we compute the first pdf of the population density, $N(t)$, for a fixed $t > 0$, in (1) using both the environmental carrying capacity pdf, f_K, and the initial population density pdf, f_{N_0}.

For each realization of N_0 and K, note that (1) becomes a deterministic IVP whose solution is

$$N(t) = N(t; N_0, K) = \frac{KN_0}{K\mathrm{e}^{-at} + N_0(1 - \mathrm{e}^{-at})}, \quad t \geq 0. \tag{2}$$

The distribution function of $N(t)$, for a fixed $t > 0$, is given by

$$\begin{aligned} F_N(q; t) &= \mathcal{P}(N(t) \leq q) = \mathcal{P}\left(\frac{KN_0}{K\mathrm{e}^{-at} + N_0[1 - \mathrm{e}^{-at}]} \leq q,\ N_0 > 0,\ K > 0\right) \\ &= \mathcal{P}(KN_0 - Kq\alpha - N_0 q[1-\alpha] \leq 0,\ N_0 > 0,\ K > 0) = \iint_{\Omega^N_{q,t}} f_{KN_0}(K, N_0)\, dK\, dN_0, \end{aligned} \tag{3}$$

where $q \in (0, +\infty)$, $0 < \alpha(t) = \exp(-at) < 1$, $\mathcal{P}$ denotes the probability measure, f_{KN_0} is the joint pdf of K and N_0 and $\Omega^N_{q,t}$ is the region defined by

$$\Omega^N_{q,t} : \begin{cases} KN_0 - Kq\alpha - N_0 q[1-\alpha] \leq 0, \\ K > 0,\ \ N_0 > 0. \end{cases}$$

Fig. 1 illustrates such region for $q = 2$ and $\alpha = 0.5$.

Since in this paper we treat N_0 and K as independent random variables, *i.e.*, $f_{KN_0} = f_K f_{N_0}$, $F_N(q; t)$ in (3) can be presented as

$$\begin{aligned} F_N(q; t) &= \int_0^{+\infty} \int_0^{q[1-\alpha]} f_K(K) f_{N_0}(N_0)\, dK\, dN_0 + \int_{q[1-\alpha]}^{+\infty} \int_0^{\frac{Kq\alpha}{K - q[1-\alpha]}} f_K(K) f_{N_0}(N_0)\, dN_0\, dK \\ &= F_K(q[1-\alpha]) + \int_{q[1-\alpha]}^{+\infty} F_{N_0}\left(\frac{Kq\alpha}{K - q[1-\alpha]}\right) f_K(K) dK, \end{aligned} \tag{4}$$