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Simultaneous determination of time and space-dependent coefficients in a parabolic equation

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ABSTRACT

This paper investigates a couple of inverse problems of simultaneously determining time and space dependent coefficients in the parabolic heat equation using initial and boundary conditions of the direct problem and overdetermination conditions. The measurement data represented by these overdetermination conditions ensure that these inverse problems have unique solutions. However, the problems are still ill-posed since small errors in the input data cause large errors in the output solution. To overcome this instability we employ the Tikhonov regularization method. The finite-difference method (FDM) is employed as a direct solver which is fed iteratively in a nonlinear minimization routine. Both exact and noisy data are inverted. Numerical results for a few benchmark test examples are presented, discussed and assessed with respect to the FDM mesh size discretization, the level of noise with which the input data is contaminated, and the chosen regularization parameters.

1. Introduction

Coefficient identification problems typically involve the estimation of certain coefficients based on inexact measurements of other measurable quantities. The estimate process is often ill-posed in the sense that small noise in the input data may lead to dramatic error in the solution. Therefore, techniques like Tikhonov regularization [24], mollification [19] and iterative regularization methods have been developed to deal with this instability [27].

Choosing an appropriate additional information about what quantities to measure or supply is important since this data enables us to identify the unknown coefficients uniquely. For instance, an upper-base final temperature condition was chosen in [4] to identify a space-dependent heat source, and a similar version can be found in [2] where a Cauchy problem for a second-order parabolic equation was formulated for determining a space-dependent coefficient of a low-order derivative. Cauchy data have also been used in [5] for reconstructing numerically a temperature-dependent thermal conductivity or a heat source. The determination of the space-dependent thermal conductivity was studied in [20] using Kansa's method based on radial basis function techniques, and in [6] using a predictor–corrector iterative finite-difference method (FDM). While spacewise dependent perfusion coefficient identification in the transient bio-heat equation subjected to time-averaging temperature measurement was investigated in [25] using the Crank–Nicolson FDM scheme combined with the first-order Tikhonov regularization method. On the other hand, time-dependent coefficient identification problems have been investigated recently, just to mention a few, the

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time-dependent inverse source identification problem [7,22,26] and the thermal conductivity/diffusivity identification problem [9,17] subjected to various kinds of overdetermination conditions.

In this paper, we consider obtaining the numerical solution of a couple of related inverse time and space-dependent coefficient identification problems in the parabolic heat equation subjected to nonlocal, time-averaging overdetermination conditions.

The organization of this paper is as follows. In Section 2, the mathematical formulations of the inverse problems are given. In Sections 3, the finite difference scheme based on the Crank–Nicholson method is developed for solving the direct problem. In Section 4, the inverse problems are reformulated as nonlinear least-squares minimization problems further penalized with Tikhonov's regularization terms in order to achieve stable solutions with respect to noise in the input data. Numerical results illustrate that accurate and stable numerical solutions are obtained, as it is discussed in Section 5. Finally, the conclusions of this research are drawn in Section 6.

2. Mathematical formulation

Let $L > 0$ and $T > 0$ be fixed numbers representing the length of a one-dimensional finite slab and the time period, respectively, and denote by $Q_T := (0, L) \times (0, T)$ the solution domain. Let also f represent a given heat source. Then consider the inverse problem of finding the time-dependent thermal conductivity $a(t)$, the space-dependent component of the fluid velocity $b(x)$ or, of the absorbtion (perfusion) coefficient $c(x)$, together with the temperature $u(x, t)$, which satisfy the parabolic heat equation

$$\frac{\partial u}{\partial t}(x,t) = a(t)\left(\frac{\partial^2 u}{\partial x^2}(x,t) + b(x)\frac{\partial u}{\partial x}(x,t) - c(x)u(x,t)\right) + f(x,t), \quad (x,t) \in Q_T, \tag{1}$$

the initial condition

$$u(x,0) = \phi(x), \quad 0 \le x \le L, \tag{2}$$

the Dirichlet boundary conditions

$$u(0,t) = \mu_1(t), \quad u(L,t) = \mu_2(t), \quad 0 \le t \le T, \tag{3}$$

the heat flux Neumann condition

$$-a(t)u_x(0,t) = \mu_3(t), \quad 0 \le t \le T. \tag{4}$$

and the time-average condition

$$\int_0^{T_0} a(t)u(x,t)dt = \psi(x), \quad 0 \le x \le L, \tag{5}$$

where $T_0 \in (0, T]$ is a given fixed number. We note that the single identifications of the coefficient $b(x)$ or $c(x)$, when $a(t)$ is known and taken to be unity, have been investigated elsewhere in [14,15].

Eq. (5) is a new overdetermination condition that in the case of heat conduction, can be regarded as the total potential heat function whose derivative, if it exists,

$$\int_0^{T_0} a(t)u_x(x,t)dt = \psi'(x), \quad 0 \le x \le L, \tag{6}$$

yields the time-average of the heat flux over the time period $[0, T_0]$. We consider therefore the following two inverse problems concerning the simultaneous determination of the coefficients $a(t)$ and $b(x)$ when $c = 0$, termed the inverse problem I, and of the coefficients $a(t)$ and $c(x)$ when $b = 0$, termed the inverse problem II. These inverse problems have been previously investigated theoretically by Ivanchov [12, Chapter 5], who establish their existence and uniqueness, as follows.

2.1. Inverse problem I

In this case $c = 0$ and Eq. (1) becomes

$$\frac{\partial u}{\partial t}(x,t) = a(t)\left(\frac{\partial^2 u}{\partial x^2}(x,t) + b(x)\frac{\partial u}{\partial x}(x,t)\right) + f(x,t), \quad (x,t) \in Q_T. \tag{7}$$

Then the inverse problem I requires determining the triplet solution $(a(t), b(x), u(x,t)) \in C[0,T] \times H^{\gamma}[0,L] \times H^{2+\gamma,1}(\overline{Q}_T)$ for some $\gamma \in (0,1)$, $a(t) > 0$ for $t \in [0,T]$, that satisfies Eqs. (2)–(4), (6) and (7). For the definition of the spaces involved, see [16]. In particular, $H^{\gamma}[0,L]$ denotes the space of Hölder continuous functions with exponent γ and $H^{2+\gamma,1}(\overline{Q}_T)$ denotes the space of continuous functions u along with their partial derivatives u_x, u_{xx}, u_t in $\overline{Q}_T$ and with u_{xx} being Hölder continuous with exponent γ in $x \in [0,L]$ uniformly with respect to $t \in [0,T]$.

Theorem 1 (Existence, see Theorem 5.2.1 of [12])**.** *Suppose that the following conditions hold:*

1. $\phi \in H^{2+\gamma}[0,L]$, $\psi \in H^{2+\gamma}[0,L]$, $\mu_i \in C^1[0,T]$ *for* $i = 1, 2$, $\mu_3 \in C[0,T]$, $f \in H^{\gamma,0}(\overline{Q}_T)$;
2. $\phi'(x) > 0$, $\psi'(x) > 0$ *for* $x \in [0,L]$, $\mu_3(t) < 0$ *for* $t \in [0,T]$;