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Short communication

Fractional order junctions

J. Tenreiro Machado *

Institute of Engineering of Polytechnic of Porto, Dept. of Electrical Engineering, Porto, Portugal

A R T I C L E I N F O

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A B S T R A C T

Gottfried Leibniz generalized the derivation and integration, extending the operators from integer up to real, or even complex, orders. It is presently recognized that the resulting models capture long term memory effects difficult to describe by classical tools. Leon Chua generalized the set of lumped electrical elements that provide the building blocks in mathematical models. His proposal of the memristor and of higher order elements broadened the scope of variables and relationships embedded in the development of models. This paper follows the two directions and proposes a new logical step, by generalizing the concept of junction. Classical junctions interconnect system elements using simple algebraic restrictions. Nevertheless, this simplistic approach may be misleading in the presence of unexpected dynamical phenomena and requires including additional "parasitic" elements. The novel γ-junction includes, as special cases, the standard series and parallel connections and allows a new degree of freedom when building models. The proposal motivates the search for experimental and real world manifestations of the abstract conjectures.

1. Introduction

Scientific progress brings together conceptual and experimental components that evolve separately. The first takes often the route of abstract thinking, producing conjectures based on philosophical and mathematical backgrounds, not immediately, or necessarily, related to the real world. The second is guided by laboratory and computer experiments, yielding observations and data that may validate, or not, the knowledge paradigms. Both components interact between themselves leading to the maintenance, or to the substitution, of the concepts and scientific tools, though the synergistic effects may occur with a considerable time difference. The outcome of technological materialization supports fundamental science and the overall result is a progressive and accelerating development of the human knowledge [1,2].

This paper follows two generalisations proposed separately in time and research area. The first started in the seventeen century with the brilliant ideas of Leibniz about the generalization of the derivative operator. The integration and differentiation of arbitrary order [3,4], or commonly Fractional Calculus (FC), remained a pure mathematical concept until the last decades, when researchers verified its ability to model memory phenomena somehow overlooked within classical calculus. Often, FC is called an "old" and "new" mathematical tool due to its duality in what concerns the considerable time elapsed between genesis and application. The second was proposed in the second half of the twentieth century by Leon Chua [5–7] who noticed a symmetry between electrical elements and variables. According to Chua, in order to preserve the symmetry of nature, besides the resistor, capacitor and inductor, a forth element, would be necessary. He denoted "memristor" the forth

* Tel.: +351 22 8340500.
E-mail address: jtm@isep.ipp.pt

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unknown lumped element. Later this abstraction was recognized to occur in the real world and new generalizations and areas of application are presently taking place.

This short debate about scientific progress and the two generalizations introduce the proposal for the concept of generalized junction. In standard practice the "obvious" junctions are the so-called series and parallel connections, easily implementable in electrical, mechanical and hydraulic systems. These two junctions may be loosely associated with the principle of conservation of energy, but that idea is somehow misleading. In fact, hidden dynamic phenomena are neglected and the validity of such modeling concept may require extra components to extend the range of their application. In abstract, we can simply say, that the series and parallel junctions impose simple algebraic restrictions between variables and elements. In this line of thought, the generalizations proposed for dynamic elements reveal that another possible direction consists of generalizing junctions and to allow them to include more complex restrictions.

This paper studies the generalization of junctions bearing in mind the concepts of FC and memristor, using as guideline electrical elements. The manuscript is organized as follows. Section two introduces the main ideas, namely fractional derivatives, their approximation, the memristor and generalized elements. Sections three discusses the concept of junction, its extension towards more complex relationships and develops several numerical experiments illustrating the corresponding properties. Finally, section four outlines the main conclusions.

2. Preliminary concepts

This section introduces the main concepts that motivate this study. Section 2.1 presents the fundamentals of Fractional Calculus (FC), Section 2.2 outlines the generalized mean and Section 2.3 discusses the higher-order circuit elements.

2.1. Fractional Calculus

FC generalizes the concept of differentiation $D^{\alpha}f(x)$ to orders $\alpha \in \mathbb{R}$ [8–12]. FC started in 1695, based on a discussion between Gottfried Leibniz and Guillaume l'Hôpital, but remained within the scope of mathematics, until recently. In the last decades FC was adopted in several scientific areas [13–16] and its application is found in control theory, physics, stochastic processes, anomalous diffusion and many other [17–23].

There are several different definitions of fractional derivatives, being the most used the Riemann–Liouville, the Grünwald–Letnikov and the Caputo formulations [24]. Moreover, the geometrical interpretation of fractional derivatives has been also the subject of alternative approaches [25–28]. Whatever the perspective, it is recognized that fractional derivatives capture the history of past events and, consequently, that fractional order systems embed the memory of the previous dynamics. This property leads to simple (fractional) models, contrary to classical (integer) models that frequently require elaborated expressions.

Using the Fourier transform and neglecting initial conditions we have the expression:

$$\mathcal{F}\{D_t^{\alpha}f(t)\} = (j\omega)^{\alpha}\mathcal{F}\{f(t)\}, \tag{1}$$

where ω and $\mathcal{F}\{\cdot\}$ denote the Fourier variable and operator, respectively, and $j = \sqrt{-1}$. Here, $(j\omega)^{\alpha}$ means $(j\omega)^{\alpha} = |\omega|^{\alpha} \exp\left(sgn(\omega)\frac{j\alpha\pi}{2}\right)$ [12].

In electrical systems was proposed the concept of fractional impedance, sometimes called "fractor". The idea occurs interpolating between the three lumped electrical elements, namely the resistor, inductor and capacitor. The same reasoning occurs in mechanical systems for the viscous damping, spring and inertia. Since these elements implement physically three consecutive (integer order) differential relations, we conceive the possibility of having other elements in between (i.e., of fractional order). In the Fourier representation a fractional α-inductor, $0 < \alpha < 1$, or a fractional β-capacitor, $0 < \beta < 1$, have impedances $Z(j\omega) = (j\omega)^{\alpha}L$, and $Z(j\omega) = \frac{1}{(j\omega)^{\beta}C}$, respectively [29–37].

2.2. Generalized approximations for fractional derivatives

The generalized mean [38] of order $\gamma \in \mathbb{R}$ (or Hölder mean) of a series of values $x_i, i = 1, \ldots, n$, is given by:

$$m_{\gamma}(x_1, \ldots, x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^{\gamma}\right)^{\frac{1}{\gamma}}, \quad \gamma \neq 0, \tag{2a}$$

$$m_{\gamma}(x_1, \ldots, x_n) = \sqrt[n]{\prod_{i=1}^{n} x_i}, \quad \gamma = 0. \tag{2b}$$

This index has several interesting properties. For example, when $\gamma = \{-1, 0, 1\}$ expression (2) reduces to the harmonic, geometric and arithmetic means, respectively. The concept appears also in the generalized *f*-mean (or Kolmogorov mean) given by: